FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 2 Round 1 Arithmetic: Factors And Multiples

1.) _______{4,5,6}__________

2.)______{23,20,13}__________

3.)______{504,488,512}__________

1. How many natural numbers N≤{100, 150, 200} have exactly 3 distinct factors? (Note: Factors must be positive.)

   Solution:
   They must be perfect squares to have an odd number of factors, so consider 1,4,9,...100. In order to have exactly 3 factors, they must be squares of prime numbers, so 4, 9, 25, 49.

2.) How many natural numbers N≤{100,80,50} are multiples of exactly two of the following numbers: 2, 3, 5?

Solution:
Must be multiples of 6, 10, or 15 but not multiples of 30
100/6 is about 16.67 so there are 16 such numbers but since we can't count 30, 60, or 90, 13 numbers are multiples of 2 and 3 but not 5. .
100/10 is 10 , so there are 10 such numbers that are multiples of 2 and 5, but we can't count 30,60, or 90, so there are 7.
1000/15 is about = 6.67 so there are 6 such numbers that are multiples of 3 and 5, but we can't count 30,60, or 90 so there are 3.
13+7+3=23.

3.) A and B are positive integers. The greatest common factor of A and B is 4. The least common multiple of A and B is {15620,14740,16060}. What is the smallest possible value of A+B?

Solution:
15620 = 1562*2*5 = 781*2*2*5 =(2^2*5*11*71).
A =(2*2)*(other prime factors of A)
B=(2*2)*(other prime factors of B)
The possibilities are
A=2*2*5, B=2*2*11*71, A = 20, B = 3124
A= 2*2*5*11, B = 2*2*71 A = 220, B = 284
A = 2*2*5*11*71, B = 2*2 A = 15620, B = 4.
The smallest possible sum is when A = 220, B = 284 A+B = 504. 220 and 284 are the smallest pair of "amicable numbers": The sum of the proper factors of 220 is 284 and the sum of the proper factors of 284 is 220.

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Match 2 Round 2 Algebra: Polynomials And Factoring

1.)______{300,200,400}________

2.) ____{23,22,21}___________

3.) _______26__________________

1.)_ Suppose that, for any value of *x*,

(4x+5)({3,2,4}x-20)-(x-4)(Ax+B)={-42x,-63x,-21x}

Find AB.

Solution:
We need 4x*3x - x*Ax = 0 and 5*(-20)-(-4*B)=0.
So A = {12,8,16}, B = 25. The middle terms check by adding to -42x. AB = {300,200,400}.

2.) For what positive value of k does x^3-7x^2+ (k^2 – {22,21,20}k)x - {26,24,22}=0 have solution x =2?

Solution:
2^3 - 7(2^2)+(k^2 – 22k)(2)-26=0, 8 - 28 + 2(k^2 – 22k) - 26 = 0,
k^2 – 22k – 23 =0(k – 23)(k + 1) = 0, **k = 23**

3.) For how many distinct integers B does $16x^2 + Bx + 81$ factor into two binomials with integer coefficients?

Solution:
We have

(16x+1)(x+81)	(16x+3)(x+27)	(16x+9)(x+9)
(8x+1)(2x+81)	(8x+3)(2x+27)	(8x+9)(2x+9)
(4x+1)(4x+81)	(4x+3)(4x+27)	(4x+9)(4x+9)
(2x+1)(8x+81)	(2x+3)(8x+27)	The other two following this
(x+1)(16x+81)	(x+3)(16x+27)	pattern are duplicates of above, and

every other combination is also a duplicate.

Including the negative integers, there are 26.

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Match 2 Round 3 Geometry: Area and Perimeter

1.____{60,30,48}______

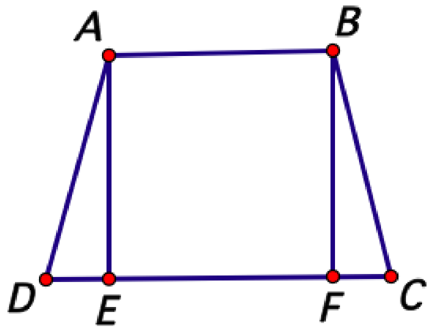
2. ____{46,60, 42}_____

3._____ {100,25,225}________

1. The length of a rectangular swimming pool is twice its width. The pool is surrounded by a sidewalk that is 3 feet wide. The area enclosed by the sidewalk and the pool is {416,176, 308} square feet. What is the perimeter of the pool? (Do not include a unit in your answer.)

Solution:
Let W=width of pool, 2W=length of pool. The width of the area enclosed by the sidewalk and pool is W+3+3 =W+6. The length of the area enclosed by the sidewalk and pool is 2W+3+3 =2W+6.
$(W+6)(2W+6)=416$, $2W^2 + 18W + 36 = 416$,
$2W^2 + 18W - 380 = 0$, $W^2 + 9W - 190 = 0$
$(W+19)(W-10)=0$, W cannot be -19, so W=10. The perimeter is
10+10+20+20 = 60 feet.

1. The trapezoid ABCD shown in the diagram is isosceles with bases AB and DC. Segments AE and BF are drawn from A and B perpendicular to segment DC. AB=5, DC={15,21,17}, and the area of rectangle ABFE is {60,75,40}. Find the perimeter of trapezoid ABCD.



Solution:
AB=5 and DC=15. Since the trapezoid is isosceles, CF=DE, so CF=DE=5 cm. The area of rectangle ABEF is 60 square cm, so BE=60/5 = 12 cm. AD=BC = 52+122 = 13, so the perimeter is 5+15+13+13 = 46 cm.

3.) ΔABC is inscribed in a circle with center O. Segment BC is a diameter of the circle. There is a number *x* such that AB = {x+5, x+4, x+6}, AC = {3x-5,3x+2,3x-12} and BO = x+3. The area of the circle is Qπ. Find Q.

Solution:
BC must be the hypotenuse of a right triangle, and O must be the midpoint of BC, so 2*(BO) = BC. By Pythagorean theorem,
$(x+5)^2 + (3x-5)^2 = (2x+6)^2$
$x^2 + 10x + 25 + 9x^2 - 30x + 25 = 4x^2 + 24x + 36$
$10x^2 - 20x + 50 = 4x^2 + 24x + 36$
$6x^2 - 44x + 14 = 0$
$3x^2 - 22x + 7 =0$
(3x-1)(x-7)=0, x=13 or x=7, but x=13 gives a negative number for AC. BO is a radius of the circle and is 7+3=10, so the area is 100π.

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Match 2 Round 4 Algebra 2: Inequalities And Absolute value

1) {69, 77, 83}__________

2.) ____{2,4,6}_________________

3.) _{_6, 4, 2}____________________

1.) How many integers satisfy the inequality below?

$$x^2 \leq \{1200, 1500, 1700\}$$

Solution:
The square root of 1200 is between 34 and 35. So the integers that satisfy the inequality are −34, −33, −32, … , −1, 0, 1, … , 34. The number of such integers is therefore 2·34 + 1 = 69.

2.) If you solve $\frac{3x-5}{x+2} > \{K, \frac{K}{2}, \frac{K}{3}\}$ for x, the solution is "x>9 or x<-2". What is K?

Solution:
If x>-2, we have 3x-5>K(x+2), so 3x-5>Kx+2K, x > (2K+5)/(3-K). x>9. (2K+5)/(3-K)=9, so (2K+5)=(27-9K), 11K = 22, K = 2. Checking the condition for x<-2 gives K=2.

3.) There are two values of K for which |x-{3,2,1}|+|x+K| = 5 has infinitely many solutions. Find the absolute value of the sum of these two values.

Solution:
In order to have infinitely many solutions, one branch of the graph of the absolute value expression |x-3| has to cancel with the opposite branch of the graph of |x+K|. Since they have to meet at y=5, -3 and K must be 5 units away from each other. K could be -8 and 2, and the sum is -6. The absolute value of the sum is 6.

FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 2 Round 5
Trigonometry:
Laws of Sine and
Cosine

Note: Drawings not necessarily drawn to scale. _

1.) {169, 9, 49}

2.) ____________{_72, 84,96}_____________

3.)_____{164, 198, 236}___________

1.) DXYZ has XY=8, YZ=8, XZ={13,12,14}. $\cos Y = -\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Solution:

By the law of cosines, $13^2 = 8^2 + 8^2 - 2 \cdot 8 \cdot 8 \cos Y$. So $\cos Y = (8^2 + 8^2 - 13^2)/(2 \cdot 8 \cdot 8) = -41/128$. So the answer to the question is 41 + 128 = 169.

2.) In ΔJKL, angle KJL is 30 degrees, angle JKL is 105 degrees, and KL = {12,14,16}$\sqrt{6}$. JK = $A\sqrt{B}$ in simplest radical form. Find AB.

Solution:
Angle JLK is 180 - 30 - 105 = 45 degrees.

$\frac{sin(30)}{12\sqrt{6}}$=$\frac{sin(45)}{JK}$, $\frac{\frac{1}{2}}{12\sqrt{6}}$=$\frac{\frac{\sqrt{2}}{2}}{JK}$, JK=12$\sqrt{12}$=24$\sqrt{3}$.

AB=24*3=72.

3.) The median from P to segment QR of ΔPQR meets segment QR at S. PQ = 6, RS=6, PS={8,9,10}. The length of segment PR is $\sqrt{A}$. Find A.

Solution:
S is the midpoint of QR, so QR=6. From ΔPQS, find the cosine of angle Q by $8^2 = 6^2 + 6^2 - 2 * 6 * 6 * cos(Q)$, cos(Q) = $\frac{-8}{-72}$=$\frac{1}{9}$. Then knowing cos(Q) of ΔPQR,

$(PR)^2$=$6^2 + 12^2 - 2 * 6 * 12(\frac{1}{9})$= 36+144-16=164. .

FAIRFIELD COUNTY MATH LEAGUE 2020-2021

Match 2 Round 6 Equations of Lines

‘

1.) {13, 41, 89}

2.) ____{13,26,39}____________

3.) ______{12,8,4}________

1.) A line is given in parametric form as $x = 2t + \frac{1}{3}$, y={4,10,16}t-$\frac{7}{3}$. If the equation of the line is expressed as y=mx+b, what is the value of $m^2 + b^2$?

Solution:
$t = \frac{x-\frac{1}{3}}{2}$, y=4($\frac{x-\frac{1}{3}}{2}$)-$\frac{7}{3}$, y=2(x-$\frac{1}{3}$)-$\frac{7}{3}$=2x-3.
So $m^2 + b^2 = 2^2 + (-3)^2 = 13$.

2.) A line of slope 0.5 intersects the parabola y=2x^2 +5x +3 at (-2,1) and (A,B). Find (8,16,24)(A+B).

Solution:
Equation of line is y-1=0.5(x+2), y=0.5x+2. Find the other point of intersection by setting 2x^2+5x+3=0.5x+2. 2x^2 + 4.5x+1=0, 4x^2 + 9x+2=0, (4x+1)(x+2)=0, so the x-coordinate of the other point of intersection is -0.25. The y-coordinate is 0.5(-0.25)+2 = 1.875. The point of intersection is (-0.25, 1.875), and 8(-0.25+1.875) = 13

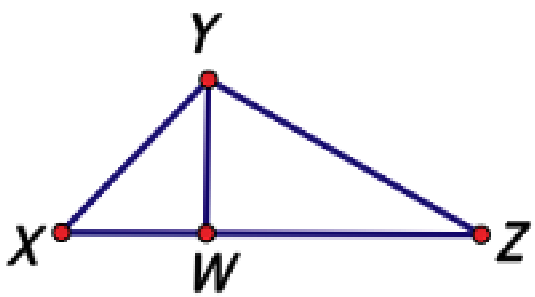
3.)_ A circle of radius 1 is centered at (0,0). The points of intersection of the circle with the perpendicular bisector of the segment whose endpoints are (2,3) and (4,-1) are (A,B) and (C,D). What is the absolute value of (30,20,10)(A+B+C+D) ?

Solution:
Midpoint of (2,3) and (4,-1) is (3,1) Slope of line connecting (2,3) and (4,-1) is -2, so desired slope is 0.5. Perpendicular bisector has equation y-1 = 0.5(x-3), or y = 0.5x-0.5. The x-coordinates of the points where this line intersects the circle are found by $x^2 + (0.5x - 0.5)^2 = 1$, $x^2 + 0.25x^2 - 0.5x + 0.25 = 1, 1.25x^2 - 0.5x - 0.75 = 0,$
$5x^2 - 2x - 3 = 0, (5x + 3)(x - 1) = 0, x = -0.6 \; or \; x = 1$. Points are (-0.6,-0.8) and (1,0). 10(-0.6+-0.8+1+0) = -4. The absolute value is 4.

FAIRFIELD COUNTY MATH LEAGUE 2020-21 Match 2 Team Round

Answers:
1. 110
2. 904
3. 234
4. 12
5. 23
6. 28

1.)_ The diagram shows ΔXYZ, in which 2*(XY)=XZ. The altitude from Y to segment XZ meets segment XZ at W and has length 12. The area of ΔXYZ is 180. The perimeter of ΔXYZ is $M+3\sqrt{N}$, where *M* and *N* are positive integers and *N* is not divisible by the square of any prime. Find M + N.



Solution:

The area is 0.5(YW)(XZ) = 0.5(12)XZ=180, SO XZ=30. XZ=2(XY), SO XY=15. Since YW=12 and ΔXYW is a right triangle, XW = 9, so angle YXW has cosine 0.6. Find YZ by $YZ^2 = XY^2 + XZ^2 - 2(XY)(XZ)\cos(\text{angle YXW})$. $YZ^2 = 30^2 + 15^2 - 2*30*15*0.6 = 585$, so YZ = $3\sqrt{65}$. Perimeter is $30 + 15 + 3\sqrt{65} = 45 + 3\sqrt{65}$. M + N = 45+65 = 110.

2.) Find the sum of the squares of all integer values of n such that $n^2 - 28n - 29$ is a prime number. (Note: Prime numbers must be positive.)

Solution:
Let $P(n) = n^2 - 28n - 29$, and note that $P(n) = (n - 29)(n + 1)$. Therefore, the only way $P(n)$ can be prime is when either $n - 29 = \pm 1$ or $n + 1 = \pm 1$, that is, when n = 30, 28, 0, or −2. $P(30) = 31$, which is a prime number. $P(28) = -29$, which is not a prime number. $P(0) = -29$, which is not a prime number. $P(-2) = 31$, which is a prime number. So, the answer to the question is $30^2 + (-2)^2 = 904$.

3.) $3x^3 + Cx^2 + Dx - 225$ factored completely over the integers is 3(x+A)(x+B)(x-B) for some values of A and B. Find the sum of all possible values of C.

Solution:
Factor a 3 out of -225 and A*B*B must be -75. The perfect square factors of 75 are 25 and 1, so A=3 and B=5 is one solution; A=75 and B=1 is another solution. $3(x+3)(x-5)(x+5) = 3x^3 + 9x^2 - 75x - 225$. $3(x+75)(x+1)(x-1) = 3x^3 + 225x^2 - 3x - 225$. 9 + 225 = 234.

4.)_ The solution to $5x^3 - 15x^2 - 20x + 72 < K$ is "$x<-2$ or $2<x<3$" Find K.

Solution:
Multiply $5(x+2)(x-2)(x-3) = 5x^3 - 15x^2 - 20x + 60$. K must be 72-60 = 12

5.) In triangle ABC, the ratio $\sin A : \sin B : \sin C$ is 5:6:7. The perimeter of the triangle is 27. The length of the longest side of the triangle is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Solution:
The Law of Sines tells us that the lengths of the sides of any triangle are in proportion to the sines of the opposite angles. So, there is a number k such that the lengths of the sides of this triangle are $5k$, $6k$, and $7k$. Since the perimeter is 27, we have $5k + 6k + 7k = 27$. This gives us $k = 3/2$, from which we get that the longest side has length $7 \cdot (3/2) = 21/2$. So the answer to the question is 21 + 2 = 23.

6.) For ΔPQR, P is at the origin, Q is at the intersection of $y = \frac{-\sqrt{3}}{3}x$ and $y = \frac{5\sqrt{3}}{3}x - 36$, and R is at the intersection of $y = \frac{\sqrt{3}}{3}x$ and $y = \frac{5\sqrt{3}}{3}x - 36$. The sine of angle PRQ is $\frac{\sqrt{a}}{b}$, where a and b are positive integers and a is not divisible by the square of any prime. Find $a + b$.

Solution:
Solve for Q by $\frac{-\sqrt{3}}{3}x$=$\frac{5\sqrt{3}}{3}x - 36$, $-\frac{6\sqrt{3}}{3}x = -36$, $-2\sqrt{3}x = -36, x = \frac{36}{2\sqrt{3}}$=6$\sqrt{3}$, y=$\frac{-\sqrt{3}}{3}$ 6$\sqrt{3}$=-6.
Solve for R by $\frac{\sqrt{3}}{3}x$=$\frac{5\sqrt{3}}{3}x - 36$, $-\frac{4\sqrt{3}}{3}x = -36$,
$x = \frac{27}{\sqrt{3}}$=9$\sqrt{3}$, y= $\frac{\sqrt{3}}{3}$ 9$\sqrt{3}$=9.
Use Law of Sines: Angle QPR is 60 degrees because each of the two lines $y = \frac{-\sqrt{3}}{3}x$ and $y = \frac{\sqrt{3}}{3}x$ is at a 30 degree angle from the origin.
Find PQ by $\sqrt{(6\sqrt{3})^2 + 6^2} = 12$
Find QR by QR=$\sqrt{(3\sqrt{3})^2 + (15)^2} = \sqrt{252}$=6$\sqrt{7}$.

$\frac{sin(angle\ PQR)}{12} = \frac{sin(60)}{6\sqrt{7}}$, $\frac{sin(angle\ PQR)}{12} = \frac{\frac{\sqrt{3}}{2}}{6\sqrt{7}} = \frac{\sqrt{3}}{12\sqrt{7}}$.

So, sin(angle PQR) = $\frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{21}}{7}$, and the answer to the question is 21 + 7 = 28.