# FCML

# 2019-20

# Season

# Match

# 4

# Teacher Packet

# FAIRFIELD COUNTY MATH LEAGUE 2019-2020

| Match 4 Round 1 |
|---|
| Arithmetic: Basic Statistics |

1.) 12

2.) 96

3.) 10

1.) A sequence of ten consecutive prime numbers has a range of 27. Find the median of this sequence.

Note that the range is odd. This means that the smallest number in the sequence must be 2; otherwise the highest and lowest values would both have been odd and the range would have been even. Therefore, the median must be the arithmetic mean of the fifth and sixth smallest prime numbers, or 11 and 13, making the answer 12.

2.) The geometric mean of $n$ numbers $a_1, a_2, \ldots, a_n$ is equal to the $n$th root of the product of the numbers: $\sqrt[n]{a_1 a_2 \ldots a_n}$. A four-term arithmetic sequence has an average (arithmetic mean) of 10 and a range of 12. The geometric mean of the numbers of this sequence can be written as $p * \sqrt[4]{q}$ where $p$ and $q$ are integers greater than 1. Find $pq$.

Let the first term be $a$ and the difference between two consecutive terms be $d$. This means the fourth term is $a + 3d$, and so $3d = 12$ and thus $d = 4$. The arithmetic mean is $a + 1.5d$, so $a = 4$ also. This makes the sequence $4, 8, 12, 16$. Therefore $\sqrt[4]{4 * 8 * 12 * 16} = \sqrt[4]{2^{11} * 3} = 2^2\sqrt[4]{2^3 * 3} = 4\sqrt[4]{24}$, so the answer is 96.

3.) A data set of 18 numbers has an average (arithmetic mean) of 70. Exactly $k$ of these numbers is 40 (where $k > 0$). If all of the numbers with a value of 40 are dropped from the data set, the average of the remaining numbers increases by a whole number. Find the number of possible values of $k$.

Let $p$ be the number of points the arithmetic mean increases by. Then we have $\frac{70(18)-40k}{18-k} = 70 + p$. This yields $p = \frac{70(18)-40k}{18-k} - 70$, and when written as a single fraction, $p = \frac{30k}{18-k}$. Considering all the integer values $k$ such that $0 < k < 18$ that make $p$ a whole number, we see that $k \in \{3,6,8,9,12,13,14,15,16,17\}$, making 10 possible values.

# FAIRFIELD COUNTY MATH LEAGUE 2019-2020

Match 4 Round 2
Algebra 1: Quadratic Equations

1.) $x = 4, x = \frac{3}{2}$

2.) $k = -4 \pm 2\sqrt{3}$

3.) $x = \frac{1}{4}$

1.) Solve for all values of : $x + \frac{12}{x} = 11 - x$.

Multiplying both sides by $x$ and bringing every term to one side yields $2x^2 - 11x + 12 = 0$, which can be factored into $(x-4)(2x-3) = 0$, giving solutions of $x = 4$ and $x = \frac{3}{2}$.

2.) If $x^2 + (k+2)x - k = 0$ has only one distinct real solution for $x$, find all possible values of $k$.

There will be only one distinct real solution for $x$ if the discriminant is equal to zero, or $(k+2)^2 - 4(-k)(1) = 0$. This gives $k^2 + 8k + 4 = 0$, which can be solved with the quadratic formula to give $k = -4 \pm 2\sqrt{3}$.

3.) Given the equation $px^2 + 3p^2 = p^3x + 3x$, $x = 6$ is one of 2 rational solutions for $x$. Find the other.

One way to solve this is to group the quadratic as $px^2 - 3x - p^3x + 3p^2 = 0 \rightarrow x(px-3) - p^2(px-3) = 0 \rightarrow (x-p^2)(px-3) = 0$, making $x = p^2$ and $x = \frac{3}{p}$ the solutions. Alternatively you can apply the quadratic formula with $a = p$, $b = -3 - p^3$, and $c = 3p^2$ to get $x = \frac{3+p^3 \pm (p^3-3)}{2p}$, which yields the same values. Therefore, either $6 = p^2$ or $6 = \frac{3}{p}$. However, if $6 = p^2$, then the other solution for $x$ would be irrational. Therefore, $6 = \frac{3}{p}$, making $p = \frac{1}{2}$ and the other rational solution for $x$ is $\frac{1}{4}$.

# FAIRFIELD COUNTY MATH LEAGUE 2019-2020

| |
|---|
| Match 4 Round 3<br>Geometry: Similarity |

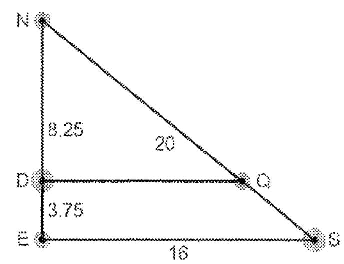
1.) 11

2.) $\frac{27}{100}$

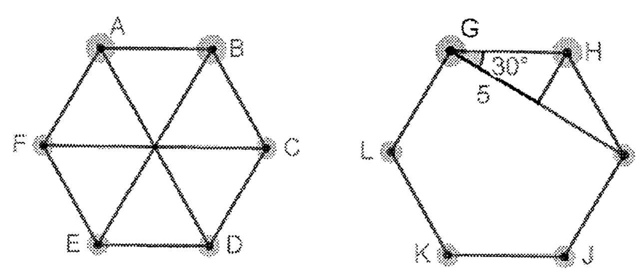
3.) $4\sqrt{3}$

1.) Consider triangle $NES$ with right angle $E$. Point $D$ lies on $\overline{NE}$ and point $Q$ lies on $\overline{NS}$ such that $\overline{DQ}||\overline{ES}$. If $NS = 20$, $NE = 12$, and $DE = 3.75$, find $DQ$.

Using Pythagorean theorem we can find that $ES = 16$. We also know that $ND = 12 - 3.75 = 8.25$. Using similarity, we can set up $\frac{8.25}{DQ} = \frac{12}{16}$, so $DQ = \frac{4}{3}\left(\frac{33}{4}\right) = 11$.

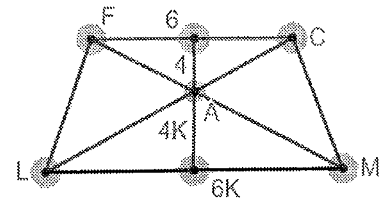



2.) Consider regular hexagons $ABCDEF$ and $GHIJKL$. If $AD = 6$ and $GI = 10$, find the value of $\frac{\text{area } ABCDEF}{\text{area } GHIJKLK}$.




Refer to the diagrams to the right. If $AD = 6$, that means that the first hexagon has side length 3. Because we can create $30-60-90$ triangles in the second hexagon, we can find that the length of one side is $\frac{10\sqrt{3}}{3}$. Since the ratio of the areas is the square of the ratio of the side lengths, we get that the ratio is $\left(\frac{3}{\frac{10\sqrt{3}}{3}}\right)^2 = \frac{81}{300} = \frac{27}{100}$.

3.) Consider trapezoid $FCML$ with bases $\overline{FC}$ and $\overline{ML}$. The diagonals of the trapezoid intersect at point $A$. $FC = 6$ and the perpendicular distance from $A$ to $\overline{FC}$ is 4. If the area of $FCML$ is 36, find the height of the trapezoid.




Refer to the diagram on the right. Since the bases are parallel, triangles $FCA$ and $MLA$ are similar. The area of the trapezoid therefore is $\frac{1}{2}(6+6k)(4+4k) = (3+3k)(4+4k) = 12(1+2k+k^2) = 36$, so $(k+1)^2 = 3$, and so $k = -1+\sqrt{3}$. Therefore the height of the trapezoid is $4 + 4(-1+\sqrt{3}) = 4\sqrt{3}$.

# FAIRFIELD COUNTY MATH LEAGUE 2019-2020

Match 4 Round 4
Algebra 2: Variation

1.) 8

2.) $c = \frac{9}{16}$

3.) $\left(\frac{12}{125}, 2000\right)$

1.) If $y$ varies directly as the square of $x$ and $y = 5$ when $x = 2$, find the positive value of $x$ when $y = 80$.

Since the variation is direct, then $\frac{y}{x^2}$ is constant. We can set up $\frac{5}{2^2} = \frac{80}{x^2}$ to get $x = 8$.

2.) Assume that $w$ varies jointly as $x$ and the square root of $y$ and inversely as the square of $z$. If $x$ is reduced to one-third its value and $z$ is reduced to one-half its value, $y$ must be multiplied by $c$ to ensure the value of $w$ remains unchanged. Find the value of $c$.

Since $\frac{wz^2}{x\sqrt{y}}$ is constant, it follows that if $\frac{wz^2}{x\sqrt{y}} = \frac{w\left(\frac{z}{2}\right)^2}{\frac{x}{3}\sqrt{cy}}$, we know $1 = \frac{\left(\frac{1}{2}\right)^2}{\frac{1}{3}\sqrt{c}}$ so $\sqrt{c} = \frac{3}{4}$, yielding $c = \frac{9}{16}$.

3.) A sphere of solid Fairfieldium has a weight that varies as the cube of its diameter and a market value that varies as the square of its weight. A sphere 1.25 cm in diameter has a weight of $\frac{3}{16}$ oz., and a sphere of diameter 2.5 cm is worth \$4500. For a sphere of solid Fairfieldium with diameter of $d$ cm, weight $w$ oz, and value $v$ dollars, find the ordered pair $\left(\frac{w}{d^3}, \frac{v}{w^2}\right)$.

We can solve directly for $\frac{w}{d^3}$ with the first ordered pair given: $\frac{\frac{3}{16}}{(\frac{5}{4})^3} = \frac{3*4^3}{16*5^3} = \frac{12}{125}$. For the second value, we can notice that the second sphere has exactly twice the diameter of the first, so its weight will be $2^3$ times greater than that of the first sphere, or $\frac{3}{2}$ oz. Finally, we can compute $\frac{4500}{\left(\frac{3}{2}\right)^2} = 2000$, making our ordered pair $\left(\frac{12}{125}, 2000\right)$.

# FAIRFIELD COUNTY MATH LEAGUE 2019-2020

| Match 2 Round 5 Precalculus: Trigonometric Expressions & DeMoivre's Theorem |
|---|

1.) $-4-4i$

2.) $\frac{-24-14\sqrt{2}}{75}$

3.) $-3$

1.) If $z=1+i$, find $z^5$ in rectangular $(a+bi)$ form.

One way to do this is to compute $((1+i)^2)^2*(1+i)=(2i)^2*(1+i)=-4(1+i)=-4-4i$. Another way is to convert $z$ to polar form. Since $z=\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i*\sin\left(\frac{\pi}{4}\right)\right)$, it follows that $z^5=\sqrt{2}^5\left(\cos\left(\frac{5\pi}{4}\right)+i*\sin\left(\frac{5\pi}{4}\right)\right)=4\sqrt{2}\left(-\frac{1}{\sqrt{2}}-i*\frac{1}{\sqrt{2}}\right)=-4-4i$.

2.) For angles $A$ and $B$ in Quadrant I, if $\cos(A)=\frac{3}{5}$ and $\sin(B)=\frac{1}{3}$, find the value of $\cos(2A+B)$.

First note that $\sin(A)=\frac{4}{5}$ and $\cos(B)=\frac{2\sqrt{2}}{3}$. Expanding this expression gives $\cos(2A)\cos(B)-\sin(2A)\sin(B)$. Employing double angle formulas gives $(2\cos^2(A)-1)\cos(B)-2\sin(A)\cos(A)\sin(B)=\left(2\left(\frac{3}{5}\right)^2-1\right)\left(\frac{2\sqrt{2}}{3}\right)-2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\left(\frac{1}{3}\right)=\left(-\frac{7}{25}\right)\left(\frac{2\sqrt{2}}{3}\right)-\frac{24}{75}=\frac{-24-14\sqrt{2}}{75}$.

3.) If $\sec(4\theta)=\frac{A\sec^4(\theta)}{B+C\tan^2(\theta)+D\tan^4(\theta)}$, where $A,B,C,$ and $D$ are relatively prime integers and $A>0$, find the value of $A+B+C+D$.

First note that $\sec(2\theta) = \frac{\sec^2(\theta)}{1-\tan^2(\theta)}$, and that $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$. If needed the first equation can be derived from $\frac{1}{\cos(2\theta)} = \frac{1}{\cos^2(\theta)-\sin^2(\theta)}$ and multiplying both the numerator and denominator by $\sec^2(\theta)$. Therefore,

$$\sec(4\theta) = \frac{\sec^2(2\theta)}{1-\tan^2(2\theta)} = \frac{\left(\frac{\sec^2(\theta)}{1-\tan^2(\theta)}\right)^2}{1-\left(\frac{2\tan(\theta)}{1-\tan^2(\theta)}\right)^2} = \frac{\sec^4(\theta)(1-\tan^2(\theta))^2}{(1-\tan^2(\theta))^2((1-\tan^2(\theta))^2-4\tan^2(\theta))} =$$

$\frac{\sec^4(\theta)}{1-6\tan^2(\theta)+\tan^4(\theta)}$, so $A + B + C + D = 1 + 1 - 6 + 1 = -3$.

# FAIRFIELD COUNTY MATH LEAGUE 2019-2020

| |
|---|
| Match 4 Round 6<br>Miscellaneous: Conic Sections |

1.) $2\sqrt{5}$

2.) $\left(-\frac{1}{2}, 6, -14\right)$

3.) $\frac{4\sqrt{30}}{3}, \sqrt{30}$

1.) Find the radius of the circle with equation $x^2 + y^2 + 8x - 4y + k = 0$ if the circle contains the point $(-2,6)$.

Looking at what would happen if we began completing the square, we notice that the center-radius form of the equation would be $(x+4)^2 + (y-2)^2 = r^2$. Substituting our point for $(x, y)$, we get $(-2+4)^2 + (6-2)^2 = 4 + 16 = 20$, so the radius is is $\sqrt{20} = 2\sqrt{5}$.

2.) For a parabola with the equation $x = ay^2 + by + c$, it is known that the distance from the vertex to the focus is equal to $|a|$, and that the focus has coordinates of $\left(\frac{7}{2}, 6\right)$ and lies to the left of the vertex. Find the ordered triple $(a, b, c)$.

Since $|a| = \frac{1}{4p}$, where $p$ is the distance to from the focus to the vertex, we can set up $a = \frac{1}{4a}$, which gives us $a = \pm\frac{1}{2}$. Since the focus is to the left of the vertex, we know that the parabola opens to the left, and thus $a = -\frac{1}{2}$. We also know that the vertex must have coordinates (4,6) so that the x-coordinate is $\frac{1}{2}$ greater than the x-coordinate of the focus. This gives an equation in vertex form of $x = -\frac{1}{2}(y-6)^2 + 4$, which when expanded and written in standard form becomes $x = -\frac{1}{2}y^2 + 6y - 14$, so the ordered triple is $\left(-\frac{1}{2}, 6, -14\right)$.

3.) An ellipse with the equation $\frac{x^2}{k^2} + \frac{y^2}{10} = 1$ has foci that are exactly $a$ units apart, where $a$ is the length of the semi-major axis. Find all possible values of the length of the horizontal axis of the ellipse.

Because the problem does not specify whether the major axis is horizontal or vertical, both cases must be handled separately. In either case, we know $r_{maj}^2 - r_{min}^2 = c^2$, where $c$ is the distance from the center of the ellipse to either focus. If the major axis is horizontal, then $r_{maj} = k$ and $c = \frac{k}{2}$, so $k^2 - 10 = \left(\frac{k}{2}\right)^2$, so $\frac{3}{4}k^2 = 10$ and $k = \frac{2\sqrt{30}}{3}$, making the horizontal axis length $\frac{4\sqrt{30}}{3}$. If the major axis is vertical, then $r_{maj} = \sqrt{10}$ and $c = \frac{\sqrt{10}}{2}$, so $10 - k^2 = \left(\frac{\sqrt{10}}{2}\right)^2$, so $k^2 = \frac{15}{2}$ and $k = \frac{\sqrt{30}}{2}$, making the horizontal axis length $\sqrt{30}$.

**FAIRFIELD COUNTY MATH LEAGUE 2019-2020 Match 4 Team Round**

1.) (70,4,14) (105,3,21)

2.) 67.5 or $\frac{135}{2}$

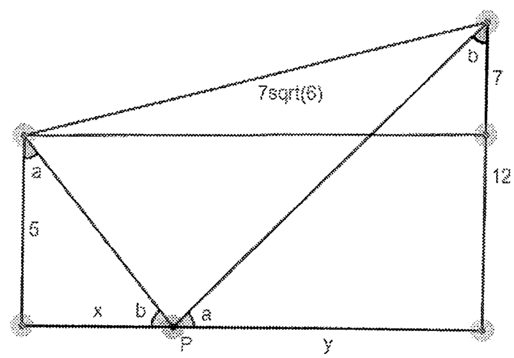
3.) $(3\sqrt{2}, \sqrt{2})$

4.) $17 + 12\sqrt{2}$

5.) $-1 + \sqrt{2}, -1 - \sqrt{2}$

6.) 8

1.) Two poles of heights 5 feet and 12 feet stand vertically upward. A rope strung tightly from the top of one pole to the top of the other has a length of $7\sqrt{6}$ feet. A point $P$ is found on the ground in between the poles such that the angles of elevation from $P$ to the tops of each pole are complementary. A second rope is strung tightly from the top of one pole to $P$ and then from $P$ to the top of the other pole. This second rope has a total length in feet of $\sqrt{a} + b\sqrt{c}$ where $a, b,$ and $c$ are integers with $a$ and $c$ having no perfect square factors greater than 1. Find all possible ordered triples $(a, b, c)$.




Consider the diagram to the right. We can solve for the distance between the bases of the poles using $(7\sqrt{6})^2 = 7^2 + d^2$, which gives $d = 7\sqrt{5}$. Using the fact that the two right triangles with the poles as legs are similar, we can set up $\frac{5}{x} = \frac{y}{12}$, so $xy = 60$, and $x + y = 7\sqrt{5}$. This gives us a quadratic $x^2 - 7\sqrt{5}x + 60 = 0$. Solving this gives $x = \frac{7\sqrt{5} \pm \sqrt{5}}{2}$, so $x = 3\sqrt{5}$ or $x = 4\sqrt{5}$. If $x = 3\sqrt{5}$ then $y = 4\sqrt{5}$ and the total length of the rope is $\sqrt{70} + 4\sqrt{14}$. If $x = 4\sqrt{5}$ then $y = 3\sqrt{5}$ and the total length of the rope is $\sqrt{105} + 3\sqrt{21}$. Therefore the two ordered triples are (70,4,14) and (105,3,21).

2.) Assume $y$ varies inversely as the $n$th power of $x$, where $n > 0$. If $y = 160$ when $x = 45$ and $y = 540$ when $x = 20$, find $y$ when $x = 80$.

Knowing that for any ordered pair $yx^n$ is constant, we know $160 * 45^n = 540 * 20^n$, which yields $\left(\frac{45}{20}\right)^n = \frac{540}{160}$, or $\left(\frac{9}{4}\right)^n = \frac{27}{8}$, so $n = \frac{3}{2}$. Since $80 = 4 * 20$, it follows that the $y$ value we seek is $\frac{540}{4^{\frac{3}{2}}} = \frac{540}{8} = 67.5$.

3.) An ellipse with the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has the same foci as the hyperbola $9x^2-7y^2=63$. If the ellipse has area $6\pi$ and the area of the ellipse is found with $\pi ab$, find the ordered pair $(a,b)$.

Note that the equation for the hyperbola can be written as $\frac{x^2}{7}-\frac{y^2}{9}=1$, which means the foci are located at $(\pm 4,0)$. This also means that $a^2-b^2=16$, since the ellipse must have its major axis be horizontal to have the same foci. We also know that $ab=6$, so $b=\frac{6}{a}$. Substitution gives us the equation $a^4-16a^2-36=0$. This factors into $(a^2-18)(a^2+2)=0$, and since $a^2\neq -2$, we know $a^2=18$ so $a=3\sqrt{2}$, which means that $b=\sqrt{2}$.

4.) The geometric mean of two numbers is found by taking the square root of the product of the numbers. For two given positive numbers $a$ and $b$ with $a>b$, the arithmetic mean is exactly three times the geometric mean. Find $\frac{a}{b}$ in simplest radical form.

Setting up the described relationship gives $\frac{a+b}{2}=3\sqrt{ab}$ or $a+b=6\sqrt{ab}$. Squaring both sides gives $a^2+2ab+b^2=36ab$, or $a^2-34ab+b^2=0$. Dividing the equation by $b^2$ gives $\left(\frac{a}{b}\right)^2-34\frac{a}{b}+1=0$, and letting $u=\frac{a}{b}$ gives $u^2-34u+1=0$. Using the quadratic formula gives $u=\frac{34\pm\sqrt{1152}}{2}=17\pm 12\sqrt{2}$. Since $a>b$, the value we are seeking must be greater than 1, so the answer is $17+12\sqrt{2}$.

5.) Consider function $f(x)=ax^2+bx+c=a(x-h)^2+k$ for nonzero $a,b,c,h$, and $k$. If $a>0$, $h=k=2c$, and $f(h+1)=-a$, find all values $p$ such that $f(p)=0$.

Setting $k=h$ and $c=\frac{h}{2}$ gives the equation $ax^2+bx+\frac{h}{2}=a(x-h)^2+h$. Expanding the square on the right side gives $ax^2-2ahx+ah^2+h$, and setting the constant terms from both sides equal gives $\frac{h}{2}=ah^2+h$. From this we know $ah^2=-\frac{h}{2}$ and so $ah=-\frac{1}{2}$ since $h\neq 0$. Also, using $f(h+1)=-a$ and vertex form gives $-a=a(h+1-h)^2+h$ or $-2a=h$. Substituting gives $a(-2a)=-\frac{1}{2}$ or $a^2=\frac{1}{4}$. Since $a>0$, we then know that $a=\frac{1}{2}$ and $h=-1$. Therefore, the equation in vertex form is $f(x)=\frac{1}{2}(x+1)^2-1$, and setting $0=\frac{1}{2}(p+1)^2-1$, we get $p=-1\pm\sqrt{2}$.

6.) For how many natural numbers $n$, $2\leq n\leq 100$, does one of the complex $n$th roots of $6-2\sqrt{3}i$ have an argument of $\frac{7\pi}{6}$?

Noting that the argument of $6 - 2\sqrt{3}i$ is $\frac{11\pi}{6}$, it follows that any of the $n$th roots would have an argument of $\frac{11\pi}{6n} + \frac{2k\pi}{n}$ for $k = 0,1,\dots,n-1$. Setting $\frac{11\pi}{6n} + \frac{2k\pi}{n} = \frac{7\pi}{6}$, multiplying everything by $6n$ and dividing a common factor of $\pi$ gives the equation $11 + 12k = 7n$, or $7n - 12k = 11$, which is a linear Diophantine equation. By inspection, we can see that the first positive integer ordered pair $(n, k)$ which solves it is $(5,2)$. It then follows that for additional points, $n$ must increase by 12 and $k$ must increase by 7. This makes all values of $n \leq 100$ that will work are 5,17,29,41,53,65,77, and 89, making 8 total values.