FAIRFIELD COUNTY MATH LEAGUE 2019-2020

Match 2 Round 1 Arithmetic: Factors & Multiples

1.) 19

2.) 56

3.) $p^8 + q^8$ or $1 + p^8 q^8$

1.) How many integers less than 100 are the product of two odd (not necessarily distinct) primes?

This set can be listed systematically: $\{3(3), 3(5), 3(7), 3(11), 3(13), 3(17), 3(19), 3(23), 3(29), 3(31), 5(5), 5(7), 5(11), 5(13), 5(17), 5(19), 7(7), 7(11), 7(13)\}$, making 19 total products.

2.) The greatest common factor of m and n is 24. The least common multiple of m and n is 360. Find the number of factors of mn.

It is known that $mn = \text{gcd}(m, n) * \text{lcm}(m, n) = 24(360)$. Alternatively students may find values of m and n that work, such as $m = 24$ and $n = 360$. Since the prime factorization of 24 is $2^3 * 3$ and the prime factorization of 360 is $2^3 * 3^2 * 5$, it follows that the prime factorization of mn is $2^6 * 3^3 * 5$. Therefore, the number of factors is $7 * 4 * 2 = 56$.

3.) Let p and q be distinct prime numbers. If the greatest common factor of A and B is pq and the least common multiple of A and B is $p^3 q^3$, find the value of $\frac{A^4+B^4}{AB}$ in terms of p and q.

There are two ways this relationship can occur. First consider the possibility that one number is not a factor of the other. Since one number must have at most one factor of p and the other at most one factor of q, but one must have p^3 as a factor and the other must have q^3 as a factor, we know that $A = p^3q$ and $B = pq^3$ (or vice versa). Therefore, $\frac{A^4+B^4}{AB} = \frac{(p^3q)^4+(pq^3)^4}{(p^3q)(pq^3)} = \frac{p^{12}q^4+p^4q^{12}}{p^4q^4} = \frac{p^4q^4(p^8+q^8)}{p^4q^4} = p^8 + q^8$. Second, consider that one factor may be a factor of the other. This is possible if $A = pq$ and $B = p^3q^3$ (or vice versa). In this case, $\frac{A^4+B^4}{AB} = \frac{(pq)^4+(p^3q^3)^4}{p^4q^4} = \frac{p^4q^4+p^{12}q^{12}}{p^4q^4} = \frac{p^4q^4(1+p^8q^8)}{p^4q^4} = 1 + p^8q^8$.

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Match 2 Round 2
Algebra 1: Polynomials and Factoring

1.) -6

2.) 4

3.) $(a+4b-2)(a+2b+2)$

1.) If $(x+1)(x-1)+x(x+2)(x-3)$ is written as ax^3+bx^2+cx+d, find the value of $a+b+c+d$.

Expanding the products makes $x^2-1+\ x^3-x^2-6x$, or x^3-6x-1, so the sum of the coefficients is -6.

2.) For how many different values of k does $x^2(x^k+3x-1)-x^4$ represent a quartic polynomial?

Note this polynomial already has a fourth degree term, so $2+k$ cannot be greater than 4. It also cannot equal 4 since then the two fourth degree terms would combine to make 0 and the polynomial would be a cubic. Therefore we have it that $0 \leq 2+k < 4$, so k could be $1, 0, -1$, or -2, giving four possibilities.

3.) Factor the following into the product of two trinomials with integer coefficients: $a^2+6ab+8b^2+4b-4$.

One way to factor this polynomial is to strategically add and subtract b^2 to get $a^2+6ab+8b^2+b^2-b^2+4b-4$ = $a^2+6ab+9b^2-(b^2-4b+$

4) $= (a + 3b)^2 - (b - 2)^2 = (a + 4b - 2)(a + 2b + 2).$

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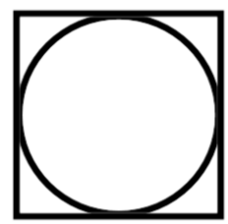
Match 2 Round 3
Geometry: Area & Perimeter

1.) $\frac{121}{8}\pi$ cm^2

2.) 1260 in^2

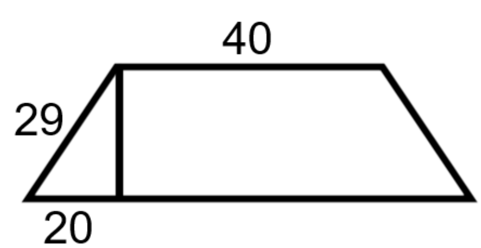
3.) 32

1.) A circle is inscribed in a square with a diagonal length of 11 cm. What is the area of the circle in square centimeters?



If the diagonal of the square is 11 cm, then it follows that a side length is $\frac{11}{\sqrt{2}}$ cm, and the radius of the circle consequently is $\frac{11}{2\sqrt{2}}$ cm, making the area $\left(\frac{11}{2\sqrt{2}}\right)^2\pi$ = $\frac{121}{8}\pi$ cm^2

2.) An isosceles trapezoid whose larger base is twice the length of its smaller base has a midsegment (median) length of 60 in and a perimeter of 178 in. What is its area in square inches?



If the smaller base is x, then the larger base is $2x$, making the median $\frac{3}{2}x$. Since $\frac{3}{2}x = 60$, it follows that $x = 40$. This means that the two bases have a combined length of 120 inches, making each of the remaining sides of the trapezoid 29 in. Because the trapezoid is isosceles, we can find the height by making a right triangle with a base of 20 and a hypotenuse length of 29 which gives us a height of 21. Therefore, the area is $21(60) = 1260 \text{ in}^2$.

3.) A rhombus has a perimeter of $24\sqrt{5}$ units and an area of 76 square units. Find the sum of the lengths of the diagonals of the rhombus.

Let half each diagonal equal a and b respectively. The area of the rhombus gives us $76 = \frac{1}{2}(d_1 d_2) = \frac{1}{2}(2a)(2b) = 2ab$. The perimeter tells us that $24\sqrt{5} = 4\sqrt{a^2 + b^2}$, so $a^2 + b^2 = 180$. Combining these two equations gives us $a^2 + 2ab + b^2 = 76 + 180 = 256$, so $(a + b)^2 = 16^2$, and therefore $a + b = 16$. Since a and b where each half of a diagonal, the sum of both of the entire diagonal lengths is 32.

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Match 2 Round 4
Algebra 2: Absolute Value & Inequalities

Remember to use AND or OR or the shorthand conjunction for a conjunction if you answer with $<$, $>$, $\leq$, or $\geq$. You may use union or intersection symbols if answering with interval notation.

1.) $x = 6$ or $x = \frac{4}{3}$

2.) $\left(\frac{11}{4}, \frac{15}{4}, \frac{23}{4}\right)$

3.) $x \in \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{2}, \frac{11}{2}\right) \cup \left(\frac{17}{3}, \infty\right)$

1.) Solve for all values of x: $|2x - 5| = x + 1$

The absolute value equation can be turned into two linear equations for x, $2x - 5 = x + 1$ and $2x - 5 = -x - 1$. Solving both equations gives solutions of $x = 6$ or $x = \frac{4}{3}$, and checking both shows neither are extraneous.

2.) The compound inequality $a < |x - b| < c$ has a solution set for x of $(-2,1) \cup \left(\frac{13}{2}, \frac{19}{2}\right)$. Write the ordered triple (a, b, c).

This compound inequality can be interpreted as 1 and $\frac{13}{2}$ being a units away from b and -2 and $\frac{19}{2}$ being c units from b. This means that b must be the exact midpoint of 1 and $\frac{13}{2}$, meaning $b = \frac{15}{4}$. Also, 1 is $\frac{11}{4}$ units from $\frac{15}{4}$, so $a = \frac{11}{4}$. Finally, -2 is $\frac{23}{4}$ units away from $\frac{15}{4}$, so $c = \frac{23}{4}$.

3.) Solve for all values of x: $\frac{1}{2|x-3|-5} < 3$.

This inequality will be satisfied if $2|x-3|-5>\frac{1}{3}$ or if $2|x-3|-5<0$. Solving the first inequality yields $|x-3|>\frac{8}{3}$, so $x\in\left(-\infty,\frac{1}{3}\right)\cup\left(\frac{17}{3},\infty\right)$. Solving the second inequality gives $|x-3|<\frac{5}{2}$, so $x\in\left(\frac{1}{2},\frac{11}{2}\right)$. Combined, the solution set is $x\in\left(-\infty,\frac{1}{3}\right)\cup\left(\frac{1}{2},\frac{11}{2}\right)\cup\left(\frac{17}{3},\infty\right)$.

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Match 2 Round 5
Precalculus: Law of Sines & Cosines

1.) $-\frac{5}{28}$

2.) $\frac{8}{5}$

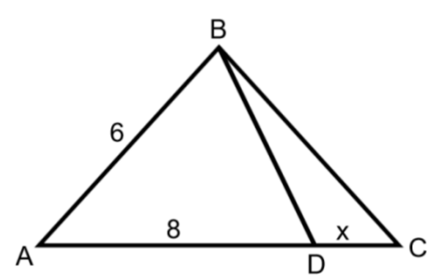
3.) $k = \frac{8}{3}$

1.) If for triangle ABC, $AB = 6$, $BC = 7$, and $AC = 10$, find the numerical value of $\min(\cos(A), \cos(B), \cos(C))$.

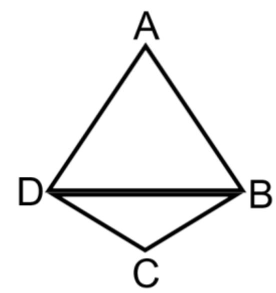
Since $6^2 + 7^2 < 10^2$, this is an obtuse triangle, and therefore the obtuse angle (B) will have the lowest cosine value. Using the law of cosines, $10^2 = 6^2 + 7^2 - 2(6)(7)\cos(B)$, so $\cos(B) = -\frac{5}{28}$.

2.) Consider triangle ABC with point D on $\overline{AC}$. If $AB = 6$, $AD = 8$, the area of triangle ABD is 10, and the area of triangle BCD is 2, find CD.

See the diagram to the right. Knowing that the area of $ABD = 10 = \frac{1}{2}(6)(8)\sin(A)$, this tells us that $\sin(A) = \frac{5}{12}$. Since the entire triangle has an area of 12, we know that $12 = \frac{1}{2}(6)(8+x)\left(\frac{5}{12}\right)$, so $x = \frac{8}{5}$.



1.) Consider kite $ABCD$ with $AB = AD = x$ and $BC = CD$. If tan (A) $= \frac{3}{4}$, $\angle A$ is supplementary to $\angle C$, and the perimeter of the kite can be written as kx, find the value of k.



Let $BC = CD = y$. One way to solve this problem is to find BD in terms of x and use that to find BC and CD in terms of x. Because tan (A) is positive, angle A is acute and therefore angle C is obtuse. We also know that $\cos(A) = \frac{4}{5}$, and so $\cos(C) = -\frac{4}{5}$. Therefore, $(BD)^2 = x^2 + x^2 - 2(x)(x)\left(\frac{4}{5}\right)$, so $(BD)^2 = \frac{2}{5}x^2$. Setting the law of cosines up for angle C gives $\frac{2}{5}x^2 = y^2 + y^2 - 2(y)(y)\left(-\frac{4}{5}\right)$, so $\frac{2}{5}x^2 = \frac{18}{5}y^2$, making $y = \frac{1}{3}x$. Therefore the perimeter is $x + x + \frac{1}{3}x + \frac{1}{3}x$, or $\frac{8}{3}x$, so $k = \frac{8}{3}$.

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Match 2 Round 6
Miscellaneous: Equations of Lines

1.) $\frac{15}{2}$

2.) $(10, -15)$

3.) $y = -\frac{2}{3}x - \frac{13}{3}$

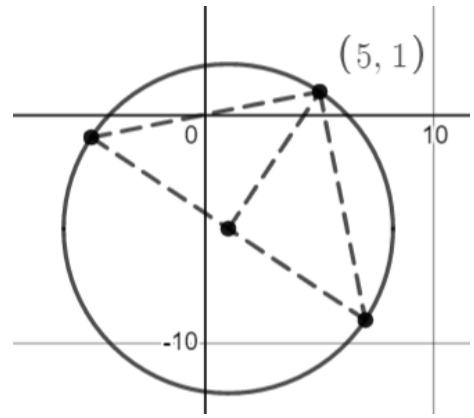
1.) If the line $Ax + By = C$ is perpendicular to $y = \frac{2}{3}x + 5$ but has the same y −intercept, find the value of $\frac{AC}{B^2}$.

One way to solve this is to write $y = -\frac{A}{B}x + \frac{C}{B}$, and then note that $-\frac{A}{B} = -\frac{3}{2}$, and so $\frac{A}{B} = \frac{3}{2}$, and $\frac{C}{B} = 5$, and so $\frac{AC}{B^2} = \left(\frac{3}{2}\right)(5) = \frac{15}{2}$.

2.) The line $y = \frac{3}{5}x + 8$ can be represented parametrically by the equations $y(t) = 6t - 1$ and $x(t) = at + b$. Find the ordered pair (a, b).

One way to solve this is to solve for t in terms of x ($t = \frac{1}{a}x - \frac{b}{a}$), and then substitute this in for t in $y(t)$ to get $y = \frac{6}{a}x - \frac{6b}{a} - 1$. A second way is to set $\frac{3}{5}x + 8 = 6t - 1$ and solve for a and b by comparing the results to $x(t)$. Finally, a third way is to know that the slope of y vs. x is found by dividing the slope of $y(t)$ by the slope of $x(t)$, making $a = 10$, and then solving for b using an arbitrary point. Letting $t = 0$ means that the point $(b, -1)$ is on the line, and so $-1 = \frac{3}{5}(b) + 8$ and therefore $b = -15$.

3.) The point (5,1) lies on the circle $(x-1)^2+(y+5)^2=52$. There are exactly two other points on the circle that are $2\sqrt{26}$ units away from (5,1). Find the equation of the line passing through these two points in slope-intercept form.



See the diagram to the right. The radius of the circle is $\sqrt{52}$. Draw isosceles triangles including the point (5,1) and the two remaining points. The length of the third side of these triangles is $2\sqrt{26}$, which is $\sqrt{2} * \sqrt{52}$. Therefore these are isosceles right triangles. Hence the line we need is perpendicular to the line passing through the center and containing $(5,1)$. The desired slope is therefore $-\frac{2}{3}$ and the line must pass through the center of the circle $(1,-5)$. The equation is then $y = -\frac{2}{3}(x-1)-5$ or $y=-\frac{2}{3}x-\frac{13}{3}$.

Team Round

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1.) 32

4.) $(x+y+z)(x-y-z)(x-y+z)(x+y-z)$

2.) $\frac{48+25\sqrt{3}}{11}$

5.) $\frac{75+25\sqrt{3}}{8}$ km

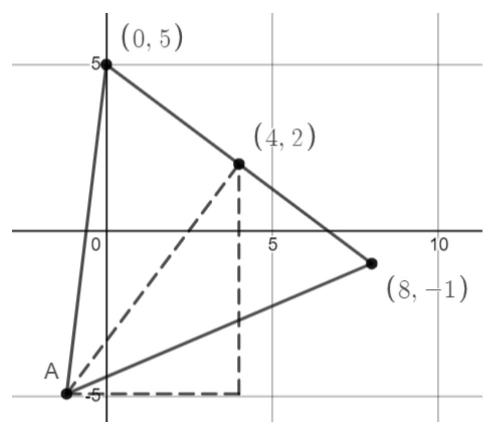
3.) $a=-b-6$

6.) 24

1.) The greatest common factor of M and 100 is 10. The least common multiple of M and 308 is 4620. Find the number of factors of the greatest possible value of M.

We know M's prime factorization contains one 2 and one 5. Since $308 = 2^2 * 7 * 11$ and $4620 = 2^2 * 3 * 5 * 7 * 11$, it follows that M's prime factorization must contain one 3 and could contain 7 and 11. Therefore the largest possible value of M has the prime factorization $2 * 3 * 5 * 7 * 11$, which has 2^5 or 32 factors.

2.) Consider triangle ABC drawn on the Cartesian plane with $AB = AC$ and with point A in Quadrant III. If point B is located at $(0,5)$, point C is located at $(8,-1)$, and the triangle has an area of $25\sqrt{3}$ square units, find the slope of the line $\overleftrightarrow{AB}$ in simplest radical form.



See the diagram to the right. Since the slope of $\overline{BC}$ is $-\frac{3}{4}$, the slope of the height of the triangle (the perpendicular bisector of $\overline{BC}$) is $\frac{4}{3}$. The height of the triangle is then the hypotenuse of a $3-4-5$ right triangle (drawn in dashes). Since $BC = 10$ and the area of the triangle is $25\sqrt{3}$, we know the length of the hypotenuse is $5\sqrt{3}$. The means the horizontal and vertical legs of the right triangle have lengths of $3\sqrt{3}$ and $4\sqrt{3}$, respectively. This provides the coordinates of A: $\left(4-3\sqrt{3}, 2-4\sqrt{3}\right)$. Finally we can determined the desired slope: $\frac{5-\left(2-4\sqrt{3}\right)}{0-\left(4-3\sqrt{3}\right)} = \frac{3+4\sqrt{3}}{-4+3\sqrt{3}} = \frac{\left(3+4\sqrt{3}\right)\left(-4-3\sqrt{3}\right)}{\left(-4+3\sqrt{3}\right)\left(-4-3\sqrt{3}\right)} = \frac{48+25\sqrt{3}}{11}$.

3.) If a and b are real numbers such that $a < 0 < b$ and the equation $|x-a| = |x-b|$ shares a solution for x with the equation $|x-b| = b+3$, find a in terms of b.

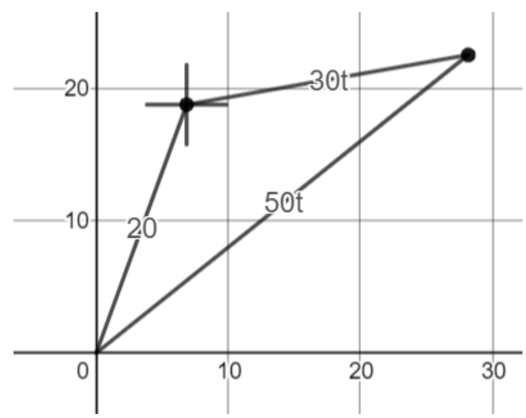
Any solution for x in the first absolute value equation would be the arithmetic mean of a

and b, or $x = \frac{a+b}{2}$. The second equation can become two linear equations in x: $x - b = b + 3$ and $x - b = -b - 3$. The solution to the first linear equation is $x = 2b + 6$. But since $b > 0$, this implies that $x > b$ which does not match the solution to the first absolute value equation. The solution to the second linear equation is $x = -3$. Therefore $\frac{a+b}{2} = -3$ and so $a = -b - 6$.

4.) Factor into trinomials with no power higher than 1: $(x^2 - y^2 - z^2)^2 - 4y^2z^2$

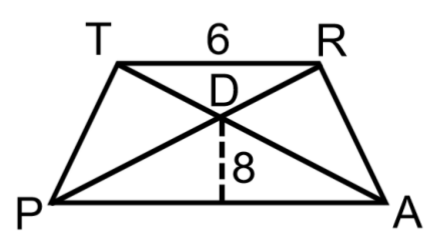
Starting first with the difference of squares: $(x^2 - y^2 - z^2 - 2yz)(x^2 - y^2 - z^2 + 2yz)$ $= \left(x^2 - (y^2 + 2yz + z^2)\right)\left(x^2 - (y^2 - 2yz + z^2)\right) = (x^2 - (y + z)^2)(x^2 - (y - z)^2) =$ $(x + y + z)(x - y - z)(x - y + z)(x + y - z)$.

5.) A cruise ship is located 20 km away from a Coast Guard outpost at a bearing of 20 degrees East of North. The cruise ship is moving at a speed of 30 km/hr at a bearing of 10 degrees North of East. If a speedboat leaves the outpost to intercept the cruise ship's course without changing direction and travels at a rate of 50 km/hr, find the total distance in kilometers traveled by the speedboat by the time it intercepts the cruise ship.



See the diagram to the right. We can set up a triangle to describe the movement of the cruise ship and the speedboat. Here we use t to represent the time it would take for the speedboat to intercept the cruise ship. Using our axes, we can determine that the angle formed from the leg representing the cruise ship's initial displacement and the leg representing the cruise ship's movement is 120^o. Using the law of cosines, we can set up $(50t)^2 = 20^2 + (30t)^2 - 2(20)(30t)\cos(120^o)$, which when expanded and simplified becomes the quadratic equation $8t^2 - 3t - 2 = 0$. The only non-extraneous solution for t is $t = \frac{3+\sqrt{73}}{16}$ hours. This means the speedboat traveled a total distance of $50t$, or $\frac{75+25\sqrt{73}}{8}$ kilometers.

6.) Consider trapezoid $TRAP$ with $\overline{TR} \parallel \overline{AP}$ and $m\angle A > m\angle P$. The diagonals intersect and point D. If $TR = 6$ and the perpendicular distance from D to $\overline{AP}$ is 8, find the area of triangle TPD.



See the diagram to the right (not necessarily drawn to scale). Let the perpendicular distance from D to $\overline{AP}$ be x and AP be y. Because triangles TRD and APD are similar, it follows that $\frac{x}{6} = \frac{8}{y}$, so $xy = 48$. Also, because triangle TPR has the same area as triangle TAR (same base length TR and same height of $8 + x$), it follows that the area of triangle TDP is equal to the area of triangle RDA. Finally, compare two ways of writing the area for the whole trapezoid: $\frac{1}{2}(b_1 + b_2)h$ and as the sum of the four smaller triangles formed from the intersecting diagonals. This gives the relationship $\frac{1}{2}(6 + y)(x + 8) = \frac{1}{2}(6)(x) + \frac{1}{2}(8)(y) + (\text{area } \Delta TDP) + (\text{area } \Delta RDA)$. Expanding and simplifying gives $3x + 4y + 24 + \frac{1}{2}xy = 3x + 4y + 2(\text{area } \Delta TDP)$, and since $xy = 48$, it follows that $48 = 2(\text{area } \Delta TDP)$, making the desired area 24.