FAIRFIELD COUNTY MATH LEAGUE 2017-2018

Match 2 Round 1 Arithmetic: Factors And Multiples

1)______217____________

2.) ________567, 1134, 2268___

3.) ______8_________________

1.)_ What is the sum of the natural number factors of 100?

$1+2+4+5+10+20+25+50+100 = 217$

2.)_ The greatest common factor of N and 3465 is 63 . The least common multiple of N and 756 is 2268. Find all possible values of N.

$63 = 3^2*7$, and $3465 = 3^2*5*7*11$ so N must have a factor of 3^2*7, but not 11, and not 5.
$756 = 2^2*3^3*7$, and $2268 = 2^2*3^4*7$. The fourth factor of 3 must come from N, so N is $3^4*7 = 567$. N could have zero 2's, one 2, or two 2's, so 567, 1134, or 2268.

3.) How many natural numbers M where $1 \leq M \leq 160000$ have exactly five natural number factors?

M must be a perfect fourth power of a prime, so we have 16 (example: factors 1,2,4,8,16}, 81, 625. Any prime p less than 20 works, so we also have 7^4, 11^4, 13^4, 15^4, and 19^4 since 20^4=160000. There are 8.

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Match 2 Round 2
Algebra: Polynomials
And Factoring

1.) ______9________________

2.) ___$-48x^3+40x^2-6x+1$

3.) _$(k^2-p)(k^2+p+1)$___

1.) For how many different whole numbers k is the expression $x^2 + 2x - k$ factorable into two binomials with integer coefficients if $0 \le k \le 100$?

k=0 factors but does not give two binomials. K must be the product of two whole numbers whose difference is two, so 1*3, 2*4, 3*5, 4*6, 5*7, 6*8, 7*9, 8*10, 9*11, and then k >100. So there are 9.

2.)_ Express as a polynomial $ax^3 + bx^2 + cx + d$ for real numbers a,b,c,d:
$1-2x(3-4x(5-6x))$

$$1-2x(3-4x(5-6x)) =$$

$$1-2x(3-(20x-24x^2) =$$

$$1-2x(3-20x+24x^2) =$$

$$1-(6x-40x^2+48x^3) =$$

$$-48x^3+40x^2-6x+1$$

3). Factor into a binomial and a trinomial: $k^4+k^2-p^2-p$

Factor by grouping:

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Match 2 Round 3
Geometry:
Area and Perimeter

1.) $12\sqrt{3}$ cm

Drawings are not necessarily to scale

2.) $\frac{1521\pi}{4}, \frac{1681\pi}{4}$ cm^2

3.) $\frac{4\pi - 3\sqrt{3}}{12\pi}$

1.) The area of a regular hexagon is $18\sqrt{3}$ cm^2. Find its perimeter.

The hexagon consists of 6 equilateral triangles of area $3\sqrt{3}$. Each triangle of side s has area $\frac{s^2\sqrt{3}}{4}$. Solve $\frac{s^2\sqrt{3}}{4} = 3\sqrt{3}$ and multiply by 6.

$$\frac{s^2\sqrt{3}}{4} = 3\sqrt{3}$$

$$s^2 = 12$$

$$s = 2\sqrt{3}$$

Perimeter is $12\sqrt{3}$ cm.

2) A right triangle with sides of integer lengths has perimeter 90 cm. A circle is circumscribed around the triangle. Give all possible areas of the circle.

The triangle must be either a 15-36-39 or 9-40-41. The diameter of the circle is equal to the hypotenuse since the inscribed angle is 90 degrees. The possible radii are $\frac{39}{2}, \frac{41}{2}$. Possible areas are $\frac{1521\pi}{4}, \frac{1681\pi}{4}$

3.) A circle of radius 6 is centered at (0,6). What fraction of the area of the circle is above the line y=9?

The total area is 36π. The line from (0,6) to (0,9) has length 3, so this makes two 30-60-90 triangles with the radii going from (0,6) to $(-3\sqrt{3},9)$ and $(3\sqrt{3},9)$. Since the central angle of the sector is 60+60=120 degrees, one-third of the area of the circle is in this sector, so to find the area of the cap, subtract the area of the 30-30-120 triangle from the area of the sector. The triangle has height 3 and base $6\sqrt{3}$ so its area is $9\sqrt{3}$. Answer is $\frac{12\pi-9\sqrt{3}}{36\pi}=\frac{4\pi-3\sqrt{3}}{12\pi}$

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Match 2 Round 4 Algebra 2: Inequalities And Absolute value

1)____ $0 < x < \frac{1}{2}$ ______

Remember to use AND or OR or the shorthand notation for a conjunction

if you answer with <. >, ≤. or ≥.

2.) ______ $\frac{3}{7}, -1$ ________

You may use union and intersection symbols if you answer using interval notation.

3.) __$x \geq \frac{1}{2}$ _or_ $x \leq \frac{-1}{6}$ _____

1.) Solve for x: $\frac{1}{x} > 2$

If x>0, multiply through by x without changing the sense of the inequality. 1>2x , so x<0.5. If x<0, then multiply through by x and change the sign, to get 2x>1, so x>0.5, which has no solutions in this domain. X can't be zero, so the solution is 0<x<0.5

2.) Find all values of x such that $|4x-1| = |2-3x|$.

If $x > \frac{2}{3}$, then |4x-1|=4x-1 and |2-3x|= 3x-2, so 4x-1=3x-2, x=-1, which is not in the domain.

If $\frac{1}{4} < x < \frac{2}{3}$, then |4x-1|=4x-1 and |3x-2|=2-3x, so 4x-1=2-3x, 7x=3, so $x = \frac{3}{7}$.

If $x < \frac{1}{4}$, then |4x-1|=1-4x, |2-3x|=2-3x, so 1-4x=2-3x. x=-1, which checks.

3.) Solve for x: $\left|5x-|x-1|\right| \geq 2$

If x≥1, then |x-1|=x-1, and 5x-(x-1)=4x+1

Solve |4x+1|≥2

4x+1≥2 or 4x+1≤-2

4x≥1 or 4x≤-3

$x \geq \frac{1}{4}\_or\_x \leq \frac{-3}{4}$.

The only part of this which fits the domain is x≥1

If x≤1, then |x-1|=1-x

5x-(1-x)=6x-1. Solve

|6x-1|≥2

6x-1≥2 or 6x-1≤-2

6x≥3 or 6x≤-1

$x \geq \frac{1}{2}\_or\_x \leq \frac{-1}{6}$, in this domain it would be $\frac{1}{2} \leq x < 1\_or\_x \leq \frac{-1}{6}$

Taken together, you get $x \geq \frac{1}{2}\_or\_x \leq \frac{-1}{6}$

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Match 2 Round 5 Trigonometry: Laws of Sine and Cosine

Note: Drawings not necessarily drawn to scale.

1.) $\frac{5\sqrt{6}}{2}$

2.) $\frac{9}{10}$

3.) $\frac{67}{112}$

1.) In ΔXYZ, the measure of $\angle Z$ is 45 degrees and the measure of $\angle X$ is 60 degrees. If XY=5, find YZ.

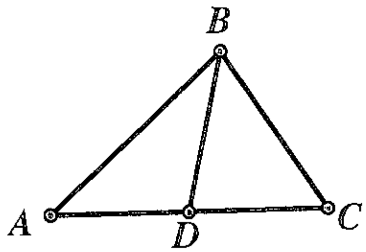
$$\frac{\sin(Z)}{XY} = \frac{\sin(X)}{YZ}$$

$$\frac{\frac{\sqrt{2}}{2}}{5} = \frac{\frac{\sqrt{3}}{2}}{YZ}$$

$$\sqrt{2} * YZ = 5\sqrt{3}$$

$$YZ = \frac{5\sqrt{3}}{\sqrt{2}} = \frac{5\sqrt{3}\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{5\sqrt{6}}{2}$$

2.) In ΔABC below, a segment from B intersects $\overline{AC}$ at D. If $\sin(\angle ACB) = \frac{2}{3}$ and $\sin(\angle CAB) = \frac{3}{5}$, find the ratio $\frac{BC}{AB}$.



For ΔBCD,

$$\frac{\sin C}{BD} = \frac{\sin D}{BC},$$

$$BC = \frac{BD * \sin D}{\frac{2}{3}} = \frac{3}{2} BD * \sin D$$

For ΔABD, the sine of angle D is the same is the sine of angle D in ΔBCD since the two angles are supplementary.

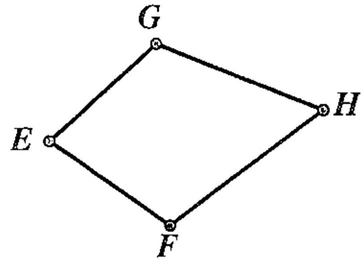
For ΔABD

$$\frac{\sin A}{BD} = \frac{\sin D}{AB},$$

$$AB = \frac{BD * \sin D}{\frac{3}{5}} = \frac{5}{3} BD * \sin D$$

$$\frac{BC}{AB} = \frac{\frac{3}{2} BD * \sin D}{\frac{5}{3} BD * \sin D} = \frac{9}{10}$$

3. In quadrilateral EFGH above, EF=5, EG=6, GH=7, and FH=8. If $\cos(\angle GEF) = \frac{1}{4}$, find $\cos(\angle GHF)$.



Draw diagonal $\overline{GF}$. $(GF)^2 = 5^2 + 6^2 - 2*5*6*\frac{1}{4} =$ 61-15=46, so

$GF = \sqrt{46}$.

$(\sqrt{46})^2 = 7^2 + 8^2 - 2*7*8*\cos(\angle GHF)$

$\cos(\angle GHF) = \frac{46-49-64}{-2*7*8} =$

$= \frac{67}{112}$

FAIRFIELD COUNTY MATH LEAGUE 2017-2018

Match 2 Round 6
Equations of Lines

1.) $y=\frac{3}{2}x-\frac{7}{2}$

2.) $y=-\frac{4}{3}x+5$

3) $\frac{81}{5}$

1.) A line is given in parametric form by $\begin{aligned} x &= \frac{1}{5}t+2 \\ y &= \frac{3}{10}t-\frac{1}{2} \end{aligned}$

Express the equation of the line in the form y=mx+b

$$x-2=\frac{1}{5}t,$$
$$t=5x-10$$
$$y=\frac{3}{10}(5x-10)-\frac{1}{2}$$
$$y=\frac{3}{2}x-3-\frac{1}{2}$$
$$y=\frac{3}{2}x-\frac{7}{2}$$

2. Give the equation of the line tangent to the circle $(x-2)^2+(y+6)^2=25$ at the point (6,-3). Express your answer in y=mx+b form.

The line from (2,-6) to (6,-3) has slope $\frac{3}{4}$, so the desired slope is $-\frac{4}{3}$ since the tangent must be perpendicular to the radius.

$$y+3=-\frac{4}{3}(x-6)$$

$$y+3=-\frac{4}{3}x+8$$

$$y=-\frac{4}{3}x+5$$

3.) Find the slope of the line connecting the centroid and the circumcenter of the triangle with vertices (2,2), (-4,6), and (6,4).

The centroid is where the medians meet. It is sufficient to find the point where two medians meet. The median from (2,2) meets the opposite side at (1,5), so it has slope -3 and y-2=-3(x-2), so y=-3x+8.

The median from (-4,6)meets the opposite side at (4,3). It has slope $-\frac{3}{8}$, so $y-3=-\frac{3}{8}(x-4)$. and $y=-\frac{3}{8}(x-4)+3, y=-\frac{3}{8}x+\frac{9}{2}$ We have

$$-3x+8=-\frac{3}{8}x+\frac{9}{2}$$

$$\frac{21}{8}x=\frac{7}{2}, x=\frac{8*7}{21*2}=\frac{4}{3}$$

y=$-3*(\frac{4}{3})+8=4$, centroid is at $(\frac{4}{3},4)$.

The circumcenter is where the perpendicular bisectors meet. The slope of the line connecting (2,2) and (-4,6) is $-\frac{2}{3}$, so we want the line with slope $\frac{3}{2}$ passing through (-1,4).

$$y-4=\frac{3}{2}(x+1)$$

$$y=\frac{3}{2}x+\frac{11}{2}$$

The slope of the line connecting (2,2) and (6,4) is $\frac{1}{2}$, so we want the line with slope -2 passing through (4,3).

$$y-3=-2(x-4)$$

$$y=-2x+11$$

$$\frac{3}{2}x+\frac{11}{2}=-2x+11$$

$$\frac{7}{2}x=\frac{11}{2},x=\frac{11}{7},y=\frac{55}{7}$$

The line connecting $(\frac{4}{3},4)\_and\_(\frac{11}{7},\frac{55}{7})$

has slope $\dfrac{\frac{55}{7}-\frac{12}{3}}{\frac{11}{7}-\frac{4}{3}}=\dfrac{\frac{165}{21}-\frac{84}{21}}{\frac{33}{21}-\frac{28}{21}}=\frac{81}{5}$

FAIRFIELD COUNTY MATH LEAGUE 2017-18 Match 2 Team Round

1.) ___80, 120, 130, 160, 170, 210______ 4.) $y=\frac{2}{3}x$______

2.) _$(a+2b)(2a+b)(2a+b)$_ 5.) ___________16+4π_________

3.) ______$3\sqrt{399}+21\sqrt{23}$ ____ 6.) _ $x<-4$ _or_ $\frac{2}{3}<x<3$ _or_ $4<x<8$ _

1.)_ M, N, and P are whole numbers, not necessarily distinct. The greatest common factor of M, N, and P is 10 and the least common multiple of M, N, and P is 100. Give all possible values of M+N+P.

Each of M, N, and P must have one factor of 2 and one factor of 5, but no more than 2 factors of 2 and 2 factors of 5.

M, N, and P could be 10, 10, and 100, M+N+P=120
M, N, and P could be 10, 20, and 100, M+N+P=130
M, N, and P could be 10, 50, and 100, M+N+P=160
M, N, and P could be 20, 50, and 100, M+N+P=170
M, N, and P could be 10, 20, and 50, M+N+P=80
M, N, and P could be 10, 100, and 100, M+N+P=210

2.) Factor into three binomials:

$$4a^3+12a^2b+9ab^2+2b^3$$

Rewrite as:

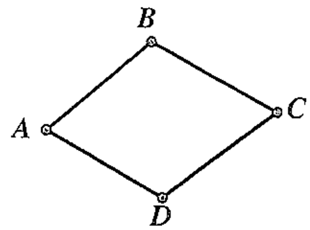
$$4a^3+12a^2b+9ab^2+2b^3$$

$$=4a^3+4a^2b+ab^2+8a^2b+8ab^2+2b^3=$$

$$a(4a^2+4ab+b^2)+2b(4a^2+4ab+b^2)$$

$$=(a+2b)(4a^2+4ab+b^2)$$

$$=(a+2b)(2a+b)(2a+b)$$



2.) Quadrilateral ABCD has $AB = 10, AD = 12, CD = 14, BC = 16,$ and $\cos(\angle A) = \frac{-1}{20}$.

Find the area of the quadrilateral.

Draw $\overline{BD}$ and find the areas of ΔABD and ΔBCD.

$$\sin(\angle A) = \sqrt{1 - (\frac{-1}{20})^2} = \sqrt{1 - \frac{1}{400}} = \sqrt{\frac{400-1}{400}} =$$

$$\frac{\sqrt{399}}{20}$$

The area of ΔABD is $\frac{1}{2} * (10)(12)\frac{\sqrt{399}}{20} = 3\sqrt{399}$

By Law of Cosines,

$$BD^2 = (10)^2 + (12)^2 - 2(10)(12)(\frac{-1}{20})$$

$$BD^2 = 100 + 144 - 240(\frac{-1}{20})$$

$$BD^2 = 244 - (-12) = 256$$

$$BD = 16$$

Use the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ to find the area. s= $\frac{16+16+14}{2} = 23$

$$A = \sqrt{23(7)(7)(9)} = 7 * 3\sqrt{23} = 21\sqrt{23}$$

Total area is $3\sqrt{399} + 21\sqrt{23}$.

4.) Give the equation of the line that connects the two intersection points of $y = |x - |5 - 2x||$ and $y = 4\left|x - \frac{5}{2}\right|$. Express your answer in y=mx+b form.

$$y = |x - |5 - 2x|| = |x - (2x - 5)| = |-x + 5| = 5 - x$$

If x>2.5,

$$y = 4\left|x - \frac{5}{2}\right|, y = 4(x - 2.5) = 4x - 10$$

Solve 5-x=4x-10, 5x=15, x=3, y=4*3-10=2. x=3, y=2

If $\frac{5}{3} < x < \frac{5}{2}$, then

$$y-2=\frac{2}{3}(x-3)$$

$$y=\frac{2}{3}x$$

Solve 3x-5=10-4x, 7x=15,

$$x=\frac{15}{7}, y=3*\frac{15}{7}-5=\frac{10}{7}$$

$$x=\frac{15}{7}, y=\frac{10}{7}$$

If $x < \frac{5}{3}$, then

$$y=|x-|5-2x||=|x-(5-2x)|=|3x-5|=5-3x$$

$$y=4\left|x-\frac{5}{2}\right|, y=4(2.5-x)=10-4x$$

Solve 5-3x=10-4x, x=5 which is not in the domain, so no possible answers here.

Slope of line between two solutions is $\dfrac{\frac{10}{7}-2}{\frac{15}{7}-3}=\dfrac{\frac{-4}{7}}{\frac{-6}{7}}=\dfrac{2}{3}$

Equation is

$$y-2=\frac{2}{3}(x-3)$$

$$y=\frac{2}{3}x$$

5. Give the perimeter of the region bounded by the system of inequalities

$$x^2-2x+y^2+6y\le 54$$

$$2x+5y\ge -13$$

$$5x-2y\le 11$$

$x^2-2x+y^2+6y\le 54$ is a circle, find its center and radius

$$x^2-2x+y^2+6y\le 54$$

$$x^2-2x+1+y^2+6y+9\le 54+10$$

$$(x-1)^2+(y+3)^2\le 64$$

so this has center (1,-3) and radius 8. Since this is ≤, this region is the interior of the circle. Both of the lines pass through (1,-3) so they both contain diameters of the circle and the two lines intersect at (1,-3). The two lines are also perpendicular to each other and the direction of the inequalities is such that together they cut off a 90 degree central angle of the circle, so the perimeter of the region is two radii plus one fourth of the circumference of the circle 2*π*8, so 8+8+4π = 16+4π.

6. Solve for x: $\dfrac{5}{x^2-7x+12} > \dfrac{3}{x+4}$

If x>4. Both denominators are positive, so $5(x+4)>3(x^2-7x+12)$, $5x+20>3x^2-21x+36$, $3x^2-26x+16<0$. This factors to (3x-2)(x-8)<0, true when x is between $\frac{2}{3}$ and 8, so we have 4<x<8.

If 3<x<4, the first denominator is negative, so $5x+20<3x^2-21x+36$, $3x^2-26x+16>0$, true when x is $x<\frac{2}{3}$ or x > 8, so no solutions in this domain.

If -4<x<3, both denominators are positive, so again x is between $\frac{2}{3}$ and 8, and this overlaps the domain for $\frac{2}{3}<x<3$.

If x<-4, the second denominator is negative, so $3x^2-26x+16<0$, true when $x<\frac{2}{3}$ or x > 8, so we also have x<-4.

Total solution: $x<-4\_or\_\frac{2}{3}<x<3\_or\_4<x<8$