FAIRFIELD COUNTY MATH LEAGUE 2016-2017

Match 2 Round 1
Arithmetic: Factors
And Multiples

1)______267____________

2.) __________70________

3.) ____88, 264, 792_____

1.) How many natural numbers are in the set {positive integers less than 1000 which are multiples of 3 but not multiples of 15} ?

The multiples of 3 are {3,6,9,…999}, so there are 333 of them. Every fifth one is a multiple of 15. The last multiple of 15 is 990, so take four-fifths of the 330 multiples of 3 up to 190, and add 3 more for 993, 996, 999. 0.8(330)+3 = 264+3 = 267.

2.) How many natural numbers M where $1 \le M \le 100$ have at least 4 distinct natural number factors?

Eliminate 1, all the prime numbers (2 factors) and all the numbers that are squares of prime numbers (3 factors).
Take 100 and subtract 1, subtract the number of primes (25) and subtract the 4 numbers that are squares of primes (4, 9, 25, 49) to get 70.

3.)_The greatest common factor of N and 2420 is 44 . The least common multiple of N and 99 is 792. Find all possible values of N.

By the test of divisibility for 11, everything has a factor of 11, so
$44 = 2^2*11$, and $2420 = 2^2*5*11^2$, so N must have a factor 2^2*11, but not 11^2, and not 5.
$99 = 3^2*11$, and $792 = 2^3*3^2*11$. The third factor of 2 must come from N, so N is at least $2^3*11 = 88$. N could have no factors of 3, one factor of 3, or two factors of 3, so N could be 88, 264, or 792.

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Match 2 Round 2
Algebra: Polynomials
And Factoring

1.) __________9____________

2.) _________225_____________

3.) _$(a^2+1)(2a-5)(a-1)$___

1.) NEW: For how many different integers k is the expression $x^2 + kx - 36$ factorable into two binomials with integer coefficients?

Each value of k can be obtained by taking two factors of 36, but you need one positive and one negative, and each combination of factors can be placed either with the larger number with the positive or the larger number with the negative.
(x+1)(x-36), (x+2)(x-18), (x+3)(x-12), (x+4)(x-9), (x+6)(x-6), (x+9)(x-4), (x+12)(x-3), (x+18)(x-2), and (x+36)(x-1) will all give different values of k, so there are 9.

2). NEW: What is the coefficient of x^2 in the expansion of (x+1)(x+2)(x+3)(x+4)(x+5) ?

$$(x+1)(x+2)(x+3)(x+4)(x+5)=$$

$$(x^2+3x+2)(x+3)(x^2+9x+20)=$$

$$(x^3+6x^2+11x+6)(x^2+9x+20)$$

The x^2 terms will come from

$$11x*9x+6x^2*20+6*x^2=99x^2+120x^2+6x^2=$$

$$225x^2$$

so the coefficient is 225.

3). NEW: Factor into three binomials with integer coefficients:

$2a^4 - 7a^3 + 7a^2 - 7a + 5$

$2a^4 - 7a^3 + 7a^2 - 7a + 5 =$

$2a^4 - 7a^3 + 5a^2 + 2a^2 - 7a + 5 =$

$a^2(2a^2 - 7a + 5) + 1(2a^2 - 7a + 5)$

$= (a^2 + 1)(2a - 5)(a - 1)$

FAIRFIELD COUNTY MATH LEAGUE 2016-2017

Match 2 Round 3
Geometry:
Area and Perimeter

1.)________ $\frac{27\sqrt{3}}{2}$ _____________cm2

Drawings are not necessarily to scale

2.) ___________56_______________cm_

3.) _____$18+12\sqrt{2}$ _______________cm

1.) The perimeter of a regular hexagon is 18 cm. Find its area.

The hexagon consists of 6 equilateral triangles of side 3. Multiply

$$6*\frac{3^2\sqrt{3}}{4}=\frac{27\sqrt{3}}{2}$$

2) The hypotenuse of a right triangle is 6 less than the sum of the legs. The area of the triangle is 84 cm2. Find the perimeter of the triangle.

If c is the hypotenuse, and a and b are the bases, we have:

$$a^2+b^2=c^2,(c+6)=a+b,\frac{1}{2}ab=84$$

Square both sides of a+b=c+6

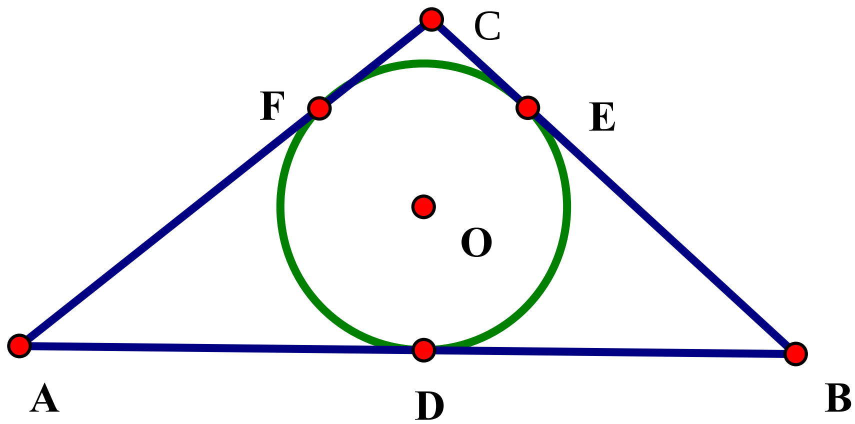
$$a^2+2ab+b^2=c^2+12c+36$$

$a^2+b^2=c^2$, so $2ab=12c+36$. If $\frac{1}{2}ab=84$, then $2ab=336$, so

$336=12c+36, 12c=300, c=25.$ $c+6=a+b,\_so\_31=a+b$

$a+b+c=31+25=56$

3.) A circle is inscribed in isosceles right triangle ABC with the right angle at C as shown. If the area of the circle is $9p$ cm2 and the tangent of 22.5 degrees is $\sqrt{2}-1$, find the perimeter of the triangle.



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Let the circle have center O. Let ΔABC be the triangle with the right angle at C. $\overline{AB}$ intersects the circle at D, $\overline{BC}$ intersects the circle at E, and $\overline{AC}$ intersects the circle at F. The radius of the circle is 3, since $\pi r^2 = 9\pi$. Quadrilateral CEOF must be a square, since it has 3 right angles and 2 congruent adjacent sides, so CE and CF are both 3. AD and AF must be equal since they are both tangent to the circle, and must have the same length as BE and BD. Quadrilateral ADOF has two right angles and a 45 degree angle at A. Since it is a kite, the diagonals bisect the angles. Draw the diagonal from A to O to make right triangle AOE with angle A = 22.5 degrees. Tan(A) = $\frac{OE}{AE} = \sqrt{2}-1$.

$OE = 3$, so $AE = \frac{3}{\sqrt{2}-1} = \frac{3(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = 3\sqrt{2}+3$

Then AC = AE+CE = $3\sqrt{2}+6$. BC also must have length $3\sqrt{2}+6$.
AB is $\sqrt{2}(3\sqrt{2}+6) = 6+6\sqrt{2}$, so the perimeter is
$2(3\sqrt{2}+6)+(6+6\sqrt{2}) = 18+12\sqrt{2}$

FAIRFIELD COUNTY MATH LEAGUE 2016-2017

Match 2 Round 4
Algebra 2: Inequalities
And Absolute value

1)____ $-1 \le x \le 6$ _____

Remember to use AND or OR or the shorthand notation for a conjunction if you answer with <. >, ≤. or ≥ . You may use union and intersection symbols if you answer using interval notation.

2.) __ -2, 2, 4, 8 ________

3.) __$x > \frac{5}{2}$_or_$x < -5$_____

1.) Solve for x: $\left|5-2x\right| \le 7$

$\left|5-2x\right| \le 7$

$-7 \le 5-2x \le 7$

$-12 \le -2x \le 2$

$6 \ge x \ge -1$

$-1 \le x \le 6$

2.) Find all values of x such that $\left|3-\left|x-3\right|\right| = 2$.

If x>3, then |x-3|=(x-3), so

$\left|3-(x-3)\right| = 2, \left|6-x\right| = 2, x = 8$_or_$x = 4$,

If x<3, then |x-3|=3-x, so

$\left|3-(3-x)\right| = 2, \left|-x\right| = 2, x = 2$_or_$x = -2$

If x-2x+3=1, then –x=-2 and x=2. If x-(2x-3)=-1, then –x+3=-1, and –x=-4, and x=4.

3.) Solve for x: $0 < \frac{x+5}{x} < 3$

$$0 < \frac{x+5}{x}\_and\_\frac{x+5}{x} < 3$$

$$If\_x > 0, then$$

$$x+5 > 0\_and\_x+5 < 3x$$

$$x > -5\_and\_x > \frac{5}{2}, so\_x > \frac{5}{2}$$

$$If\_x < 0, then$$

$$x+5 < 0\_and\_x+5 > 3x$$

$$x < -5\_and\_x < \frac{5}{2}, so\_x < -5$$

$$Solution: x > \frac{5}{2}\_or\_x < -5$$

FAIRFIELD COUNTY MATH LEAGUE 2016-2017

Match 2 Round 5
Trigonometry:
Laws of Sine and Cosine

Note: Drawings not necessarily drawn to scale. _

1.) ______________ $\frac{3\sqrt{2}}{5}$ ________________

2.) _____________0.52_________________

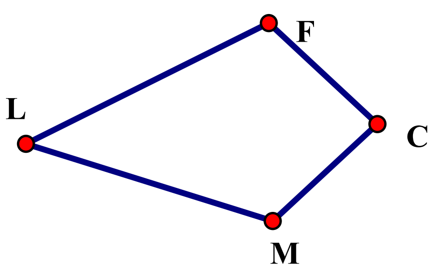
3.)________________10 _____________

1.) In ΔXYZ, the measure of ÐZ is 45 degrees. YZ=6 and XY=5. Find the sine of ÐX.

$$\frac{\sin(Z)}{XY} = \frac{\sin(X)}{YZ}$$

$$\frac{\frac{\sqrt{2}}{2}}{5} = \frac{\sin(X)}{6}$$

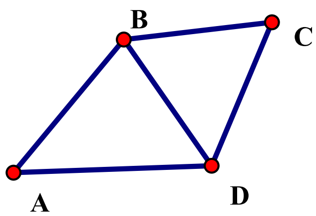
$$\sin(X) = \frac{6*\frac{\sqrt{2}}{2}}{5} = \frac{3\sqrt{2}}{5}$$



2.)_In kite FCML above, FC=CM=8, FL=ML=10, and cos(ÐC)=0.25. Find cos(ÐL).

Draw diagonal $\overline{FM}$.

$$(FM)^2 = (FC)^2 + (CM)^2 - 2 * FC * CM * \cos(\angle C)$$
$$(FM)^2 = 8^2 + 8^2 - 2 * 8 * 8 * 0.25$$
$$(FM)^2 = 128 - 32 = 96$$
Also
$$(FM)^2 = (FL)^2 + (CL)^2 - 2 * FL * CL * \cos(\angle L)$$
$$96 = 10^2 + 10^2 - 2 * 10 * 10 * \cos(\angle L)$$
$$96 = 200 - 200\cos(\angle L)$$
$$-104 = -200\cos(\angle L)$$
$$\cos(\angle L) = \frac{104}{200} = 0.52$$



3. Diagonal $\overline{BD}$ is drawn in quadrilateral ABCD as above.. AB=15, BC=12. $\sin(\angle BAD)$=0.7, $\sin(\angle BDA)$=0.75, and $\sin(\angle BCD) = \frac{2\sqrt{6}}{5}$. $\angle BAD$, $\angle BDA$, and $\angle BCD$ are all acute angles. Find CD.

By law of sines,
$$\frac{\sin(\angle BAD)}{BD} = \frac{\sin(\angle BDA)}{AB}$$
$$\frac{\frac{7}{10}}{BD} = \frac{\frac{3}{4}}{15}, \frac{3}{4} * BD = \frac{7}{10} * 15, \frac{3}{4} * BD = 10.5,$$
$$BD = \frac{4}{3} * 10.5 = 14$$

$$\cos(\angle BCD) = \sqrt{1 - \sin^2(\angle BCD)} = \sqrt{1 - (\frac{(2\sqrt{6})}{5})^2} = \sqrt{1 - \frac{24}{25}} = \sqrt{\frac{1}{25}} = \frac{1}{5}$$, , take the positive value since the angle is acute. By law of cosines,

$$(BD)^2 = (BC)^2 + (CD)^2 - 2 * BC * CD * \cos(\angle BCD)$$

$$14^2 = 12^2 + (CD)^2 - 2 * 12 * (CD) * \frac{1}{5}$$

$$(BD)^2 = (BC)^2 + (CD)^2 - 2 * BC * CD * \cos(\angle BCD)$$

$$14^2 = 12^2 + (CD)^2 - 2 * 12 * (CD) * \frac{1}{5}$$

$$196 = 144 + (CD)^2 - \frac{24}{5} CD$$

$\text{Re}\, arrange$

$$CD^2 - \frac{24}{5} CD - 52 = 0$$

$Multiply\_by\_5$

$$5(CD)^2 - 24CD - 260 = 0$$

$Factor$

$$((CD) - 10)(5(CD) + 26) = 0$$

$Solve$

$$CD = 10$$

since we can’t take the negative answer.

FAIRFIELD COUNTY MATH LEAGUE 2016-2017

Match 2 Round 6
Equations of Lines

1.) ___$y = \frac{3}{2}x + \frac{11}{2}$________

2.) ____$y = \frac{-3}{4}x + \frac{25}{2}$____________

3) ____________$\frac{5}{2}$___________

1.) A line is given in parametric form by $\begin{array}{l} x = 2t - 1 \\ y = 3t + 4 \end{array}$

Express the equation of the line in the form y=mx+b

$$t = \frac{x+1}{2}$$

$$y = 3\left(\frac{x+1}{2}\right) + 4$$

$$y = \frac{3}{2}x + \frac{3}{2} + 4$$

$$y = \frac{3}{2}x + \frac{11}{2}$$

2. Find the equation of the line passes through the point of tangency of of the circles $x^2 + y^2 = 100$ and $(x-9)^2 + (y-12)^2 = 25$ and is tangent to both circles. Express your answer as y=mx+b.

Solve for the point of tangency of the two circles: The line through the center of the circles must have y-intercept 0 since it passes through the origin, and slope $\frac{12-0}{9-0}=\frac{4}{3}$, so $y=\frac{4}{3}x$. The point of tangency must be 10 from the origin along this line, so

$$\sqrt{x^2+(\frac{4}{3}x)^2}=10$$
$$\sqrt{x^2+\frac{16}{9}x^2}=10$$
$$\sqrt{\frac{25}{9}x^2}=10$$
$$\frac{5}{3}|x|=10$$
$$x=6$$

since it must be 10 units along the line towards (9,12), so x is not -6, x=6, y= $\frac{4}{3}x$, so y=8. The point of tangency is (6,8)

Solve for the equation of the line.

$$\sqrt{x^2+(\frac{4}{3}x)^2}=10$$
$$\sqrt{x^2+\frac{16}{9}x^2}=10$$
$$\sqrt{\frac{25}{9}x^2}=10$$

The line must be perpendicular to the line containing

$$\frac{5}{3}|x|=10$$
$$x=6$$

(0,0) and (6,8), so the slope of the line must be $\frac{-3}{4}$.

$$y - 8 = \frac{-3}{4}(x - 6)$$

$$y - 8 = \frac{-3}{4}x + \frac{9}{2}$$

$$y = \frac{-3}{4}x + \frac{25}{2}$$

3.) x+2y=5 is the perpendicular bisector of the line segment whose endpoints are $(k^2 - \frac{37}{4}, k - 1)$ and $(2k - 6, -3k + 13)$ Find all possible values of k.

x+2y=6 has slope $\frac{-1}{2}$, so we need the slope of the line connecting the two points to be 2.

$$\frac{-3k + 13 - (k - 1)}{2k - 6 - (k^2 - \frac{37}{4})} = 2, \frac{-4k + 14}{-k^2 + 2k + \frac{13}{4}} = 2,$$

$$-4k + 14 = -2k^2 + 4k + \frac{13}{2}$$

$$-8k + 28 = -4k^2 + 8k + 13$$

$$4k^2 - 16k + 15 = 0$$

$$(2k - 5)(2k - 3) = 0$$

$$k = \frac{5}{2}, k = \frac{3}{2}$$

Check if either value gives a bisector of the segment: If $k = \frac{5}{2}$, the endpoints are $(-3, \frac{3}{2})\_and\_(-1, \frac{11}{2})$ with midpoint $(-2, \frac{7}{2})$, and x+2y=5 goes through the midpoint. If $k = \frac{3}{2}$, then the endpoints are

$(-7, -3)\_and\_(\frac{1}{2}, \frac{17}{2})$, with midpoint (-3.25, 2.75), and x+2y=5 does not intersect this point, so the $k = \frac{3}{2}$ does not meet the bisector condition, so the only solution is $k = \frac{5}{2}$

FAIRFIELD COUNTY MATH LEAGUE 2016-17 Match 2 Team Round

1.) _____10416___________

4.) $-7 < x < -5$ _or_ $-3 < x < -1$______

2.) _$(a+2b)(a^2+3+2b)(a^2+3-2b)$_

5.) ____________$\frac{16}{19}$_________

3.) ______343, 675, 768, 800____

6.) ____________y = x-4_______________

1.) 496 is a called a perfect number, because the sum of its proper divisors 1+2+4+8+16 +31+62+124+248=496. The only two perfect numbers less than 496 are both less than 30. Find the least common multiple of the three smallest perfect numbers.

The other two perfect numbers are 6 = 1+2+3 and 28=1+2+4+7+14.
6=2*3, 28=2^2*7, 496=2^4*31. LCM is 2^4*3*7*31= 496*21= 10416.

2.) Factor into a binomial and two trinomials:

$$a^5+2a^4b+6a^3+12a^2b-4ab^2-8b^3+9a+18b$$

Rewrite as:

$$a^5+2a^4b+6a^3+12a^2b-4ab^2-8b^3+9a+18b$$

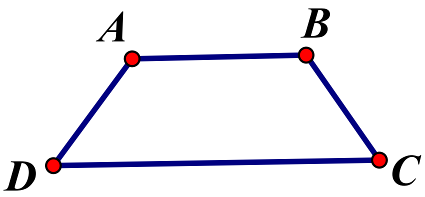
$$a^5+6a^3+9a-4ab^2+2a^4b+12a^2b+18b-8b^3=$$

$$a(a^4+6a^2+9-4b^2)+2b(a^4+6a^2+9-4b^2)=$$

$$(a+2b)(a^4+6a^2+9-4b^2)=$$

$$(a+2b)((a^2+3)^2-4b^2)=$$

$$(a+2b)(a^2+3+2b)(a^2+3-2b)$$



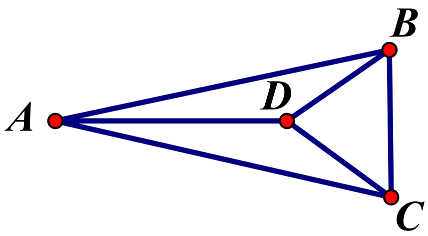
3.) ABCD is an isosceles trapezoid with bases $\overline{AB}$ and $\overline{CD}$ as the bases. The shorter base is $\overline{AB}$, and $AB = AD = BC = 25$ cm. The height and perimeter of the trapezoid are both integers. Find all possible values for the area of ABCD.

Let CD=25+2x. AD and BC must be the hypotenuses of right triangles. Since the height is an integer and the perimeter is an integer, 2x must be an integer. If x is half of an integer, there are no integer values y such that $x^2 + y^2 = 625$, so x must be an integer, and therefore the triangles must have lengths that are Pythagorean triples, so the possibilities are 15-20-25 and 7-24-25, If 25+2x is the length of the longer base, we could either have
Height = 15, x=20, area = (0.5)(65+25)(15)=45*15=675
Height = 20, x=15, area = (0.5)(55+25)(20)=40*20=800
Height = 7, x=24, area = (0.5)(73+25)(7)=49*7=343
Height = 24, x =7, area = (0.5)(25+39)(24)=32*24=768

4.) Solve for x: $|x+2| - |x+4| + |x+6| < 3$

$$|x+2| - |x+4| + |x+6| < 3$$
$$If\_x > -2\_then\_|x+2| = x+2, |x+4| = x+4, |x+6| = x+6$$
$$x+2-(x+4)+x+6<3$$
$$x+4<3, x<-1, so\_-2<x<-1$$
$$If\_-4<x<-2, then\_|x+2| = -x-2, |x+4| = x+4, |x+6| = x+6$$
$$-x-2-(x+4)+x+6<3$$
$$-x<3, x>-3, so\_-3<x<-2$$
$$If\_-6<x<-4, then\_|x+2| = -x-2, |x+4| = -x-4, |x+6| = x+6$$
$$-x-2-(-x-4)+x+6<3, x+8<3, x<-5, so$$
$$-5<x<-4$$
$$If\_x<-6, then\_|x+2| = -x-2, |x+4| = -x-4, |x+6| = -x-6$$
$$-x-2-(-x-4)-x-6<3, -x-4<3, -x<7, x>-7$$
$$so\_-7<x<-6$$

The inequality is true for x=-2 but and x=-6, but not x=-4, so the solution is
$$-7<x<-5\_or\_-3<x<-1$$



5. In the diagram above, DB=DC=10, AB=AC, AD=15, $\cos(\angle CDB) = 0.25$, and $\angle ADB = 120°$. Find $\cos(\angle CAB)$.

$$(BC)^2 = 10^2 + 10^2 - 2*10*10*(\frac{1}{4}) = 150$$

$$(AB)^2 = 10^2 + 15^2 - 2*10*15*(-\frac{1}{2}) = 475$$

$$AB = \sqrt{475} = \sqrt{25}\sqrt{19} = 5\sqrt{19}$$

so

$$(BC)^2 = (AB)^2 + (AB)^2 - 2(AB)(AB)\cos(\angle CAB)$$

$$150 = 475 + 475 - 2(5\sqrt{19})(5\sqrt{19})\cos(\angle CAB)$$

$$\cos(\angle CAB) = \frac{150 - 475 - 475}{-50*19} = \frac{-800}{-50*19} = \frac{16}{19}$$

6. ΔABC lies entirely in the first quadrant. $\overline{AB}$ lies on the line 3x – 4y = 4, and $\overline{AC}$ lies on the line 4x – 3y = 24. AB=30 and AC = 40. Find the equation of the line containing the angle bisector of $\angle A$. Express your answer as y=mx+b.

Solution: Solve the two equations simultaneously to find the A is the point (12,8). Since $\overline{AC}$ lies along a line with slope $\frac{3}{4}$, and AB=30, $\overline{AB}$ is the hypotenuse of a multiple of a 3-4-5 triangle, so to find point B from point A, move right 24 and up 18 to get to (36,26). Since $\overline{AC}$ lies along a line with slope $\frac{4}{3}$, and AC=40 , $\overline{AC}$ is the hypotenuse of a multiple of a 3-4-5 triangle, so to find point C from point A, move right 24 and up 32 to get to (36,40). Then $\overline{BC}$ lies along the line x=36 and has length 40-26=14. By the angle bisector theorem, the angle bisector splits $\overline{BC}$ into the same ratio as AB to AC. Suppose the angle bisector meets $\overline{BC}$ at point D. Then

$$\frac{AB}{BC} = \frac{BD}{CD}, \frac{30}{40} = \frac{BD}{14 - BD}, 40(BD) = 30(14 - BD),$$

$$70BD = 420, BD = 6\_and\_CD = 14 - 6 = 8$$

Then the angle bisector passes through the point (36,32). The slope of the line containing $\overline{AD}$ is $\frac{32-8}{36-12}=1$. $y-8=1(x-12), y=x-4$.