FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 3 Round 1
Arithmetic: Scientific Notation and Bases

1.) ________1, 2____________

2.) ______1.875×10^{13}__________

3.) ________35340_6______________

1 point: Find all possible values of the digit d such that $dd1_5$ is prime.

The only choices for d are 0,1,2,3,4 in base 5. If d=0, $dd1_5$ is 1, which students need to know is not prime.
If d=1, is 25+5+1 = 31, prime
If d=2 is 50+10+1 = 61, prime
If d=3 is 75 + 15 + 1 = 91 = 13x7
If d=4 is 100 + 20 + 1 = 121 = 11x11

2 points: Express the following in Scientific Notation:
$[(2 \times 10^3)^2 (3 \times 10^{-7})] / (4 \times 10^{-5})^3$

$= [(4 \times 10^6) (3 \times 10^{-7})] / (64 \times 10^{-15})$
$= [(12 \times 10^{-1})/ (64 \times 10^{-15})$
$= 12/64 \times 10^{14} = 0.1875 \times 10^{14} = 1.875 \times 10^{-1} \times 10^{14} = 1.875 \times 10^{13}$

3 points: Multiply the number 1212_3 by the number 1212_4 and express your answer as a number in base 6.

1212_3 = 27+2*9+3+2 = 50. 1212_4 is 64+2*16+4+2 = 102. 102 x 50 = 5100. To convert 5100 into base 6, 1296 goes into 5100 3 times with remainder 1212. 216 goes into 1212 5 times with remainder 132. 36 goes into 132 3 times with remainder 24. Then 6 goes into 24 4 times.

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Match 3 Round 2
Algebra: Word Problems

1.) ________9:36 AM____________

2.) _____________80_____________

3.) _______________2___________

1.) Train A leaves Norwalk at 9:00 AM and travels due east at 40 mph. Train B leaves Greenwich, which is 10 miles west of Norwalk, at 9:10 AM and travels due west at 60 mph. At what time will the two trains be 60 miles apart?

Solution: Let x = the time train A travels. Then x-1/6 is the time train B travels. The total distance must be 50, so solve 40(x) + 60(x-1/6) = 50, so 100x – 10 = 50, so 100x=60, and x=3/5 hour since train A left, so the time is 9:36 AM.

2.) Nancy has nickels, dimes, and quarters in her change purse. The total value of the money is $10. The value of the nickels and dimes combined is equal to the value of the quarters. Nancy has twice as many dimes as she has nickels. How many coins does Nancy have?

Solution:Let n=# of nickels, d=# of dimes, q = # of quarters
5n + 10d + 25q = 1000
5n+ 10d = 25q
2n=d
So substituting 2n=d into the top equations, we get 5n+ 20n + 25q = 1000 and 5n+20n=25q, so 25q+25q=1000, so q=20. Then 25q=500=5n+10d=5n+20n=
So 25n=500 and n=20. 2n=d, so d=40. The total number of coins is 20+20+40 = 80

3.) Working together, Harry, Ron, and Hermoine can clean Hagrid's stables in 1 hour. If Harry worked by himself, it would take 1.5 times as long as it would take Hermoine to do the job by herself. If Ron worked by himself, it would take 3 hours longer than it would take Harry to do the job by himself. How many hours would Hermoine need to clean Hagrid's stables by herself?

Solution: Let x=amount of time it takes Hermoine to clean the stables.
Then 1.5x = amount of time it takes Harry to clean the stables
Then 1.5x+3 = amount of time is takes Ron to clean the stables. The total time is 1 hour, so the fraction of the job each person does is 1/x, 1/(1.5x), and 1(1.5x+3), which must add to 1 stable. 1/x + 1/(1.5x) + 1/(1.5x+3) = 1, so 1.5x(1.5x+3)[1/x + 1/(1.5x) + 1/(1.5x+3)] = 1.5(1.5x+3) + 1.5x+3 + 1.5x = 1.5x(1.5x+3), so
$2.25x + 4.5 + 1.5x + 3 + 1.5x = 2.25x^2 + 4.5x$
so $2.25x^2 - 0.75x – 7.5 = 0$
Multiply by 4/3 to get $3x^2 – x – 10 = 0$, so (x-2)(3x+5) = 0, x=-5/3 is extraneous, so x=2.

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Match 3 Round 3
Geometry: Polygons

1.) ________11 ____________

2.) _____________$48\sqrt{3}$____________

3.) _______________3___________

1.) The total number of sides and diagonals of a convex polygon is 55. How many sides does the polygon have?
Solution: Solve $n + n(n-3)/2 = 55$, so $2n + n(n-3) = 110$, so $n^2 - n - 110 = 0$, $(n-11)(n+10) = 0$, and $n=-10$ is extraneous, so $n = 11$

2.) A regular hexagon has apothem 6 cm. Find the numerical difference between its area in cm^2 and its perimeter in cm.

Solution: If the apothem is 6 cm, it forms a 30-60-90 triangle with one of the diagonals and half of the one of the sides, with the apothem being $\sqrt{3}$ times the half-side, so the half-side is $6/\sqrt{3}$ cm, and the entire side is $12/\sqrt{3}$ cm. Use $A = s^2\sqrt{3}/4$ with $s=12/\sqrt{3}$ to get $A = (12/\sqrt{3})^2\sqrt{3}/4 = (144/3)(\sqrt{3}/4) = 12\sqrt{3}$ for one of the six triangles that make up the hexagon, so the total area is $72\sqrt{3}$. However, the perimeter is $6*12/\sqrt{3} = 72/\sqrt{3} = 72\sqrt{3}/3 = 24\sqrt{3}$, so the difference is $48\sqrt{3}$.

3 points: For two regular convex polygons, if you add a single interior angle from one to a single interior angle from the other, you get 306 degrees. For how many different combinations of polygons is this true? (Answer: 3 (8 and 24, 10 and 20, 12 and 15))
Solution: You need $180(n-2)/n$ to either be 2 integers or 2 fractions that conveniently add up to 306. For n=3, $180(n-2)/n = 60$ N=4, $180(n-2)/n = 90$
N=5, $180(n-2)/n = 108$ N=6, $180(n-2)/n = 120$
N=7, $180(n-2)/n$ is something over 7. N=8, $180(n-2)/n = 135$.
N=9, $180(n-2)/n = 140$ N=10, $180(n-2)/n = 144$
N=11, $180(n-2)/n$ is something over 11 N=12, $180(n-2)/n = 150$
N=13, $180(n-2)/n$ is something over 13
N=14, $180(n-2)/n$ is something over 14, but it's between 150 and 156, and $180(7-2)/7$ is between 120 and 135, so they can't add up to 306.
N=15, $180(n-2)/n = 156$, so a 12-gon and 15-gon is one combination. Running through the other combinations, you get n=20 gives interior angle 162, so 144+162 = 306, so a decagon and 20-gon is another. The final one is n=40 gives n=171, so an octagon and a 40-gon is another. So there are 3.

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Match 3 Round 4
Algebra 2: Functions and Inverses

1.) ___________ $\frac{x+93}{32}$ ________________

2.) ___ y>0 or y<-1 alternatively (-∞, -1) U (0,∞)__

3.) ______x≥-5 alternatively [-5,∞)___________

1.)If $f(x) = 2x - 3$ and $h(x) = f(f(f(f(f(x))$, find $h^{-1}(x)$. Give your answer in terms of x.

Solution: f(f(x)) = 2(2x-3)-3 = 4x-9.
f(f(f(x)) = 2(4x-9) – 3 = 8x -21
f(f(f(f(x) = 2(8x-21) – 3 = 16x – 45
f(f(f(f(f(x) = 2(16x-45) – 3 = 32x – 93. If h(x) = 32x-93, $h^{-1}(x) = (x+93)/32$

2.) $g(x) = \frac{1}{x^2 - 1}$ and $h(x) = \frac{1}{x}$. Give the range of $(g \circ h)(x)$.

Solution: g(h(x)) = $1/(1/x)^2 - 1)$ = $1/(1-x^2)/x^2$ = $x^2 / (1-x^2)$. This has asymptotes at x=1 and x=-1. If x<-1, $x^2>1$ while $(1-x^2)<1$, so the fraction must be greater than 1. As |x| increases without bound, so does g(h(x)).
If -1<x<1, you get a curve that is asymptotic to x=1 and x=-1 would reach minimum at x=0 if x=0 were in the domain, but we can't do 1/0, be we can observe that as x approaches 0, $1/(1/x)^2 -1$ approaches 0 . As x approaches -1 or -1, g(h(x)) decreases without bound.
If x>1 it's symmetric to the situation where x<-1.

Answer: { y>1 or y<-1} or (-∞, -1) U (1,∞)

3 points: $f(2x) = x^2 - 2x – 4$. Give the domain of the relation $f^{-1}(x)$.

$f(2x) = x^2 – 2x – 4$, so $f(x) = (x/2)^2 - 2(x/2) – 4 = x^2/4 – x – 4$, or

The range of this function is the domain of $f^{-1}(x)$, so if we complete the square on $x^2/4 - x-4$, we get

$((1/4)x^2 - 4x \quad) - 4$

= $¼(x^2 - 4x + 4) - 4 – 1 = (1/4)(x-2)^2 – 5$, so the range of this function is y≥-5, so the domain of the inverse is x≥-5.

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Match 3 Round 5
Advanced Math:
Exponents and Logarithms

1.) _________729_____________

2.) __________3______________

3.) ______9/2 _______________

1.) If $\log_9 a = x$, and $\log_{27} b = y$, what is the product of a and b if $2x+3y=6$?

Solution: $9^x = a$ and $27^y - b$, so $3^{2x} = a$ and $3^{3y} - b$, so $3^{2x+3y} = ab$. Since $2x+3y=6$, $ab=3^6 = 729$

2.) Solve for x: $\log_4 (9x^2 - 73) - \log_4 (3x-7) = 1$

Solution: $\log_4[(9x^2-73)/(3x-7)] = 1$, so $(9x^2-73)/(3x-7) = 4$, so $9x^2 -73 = 12x-28$
So $9x^2 - 12x - 45 = 0$, so $3x^2 - 4x - 15 = 0$, so $(3x+5)(x-3) = 0$, and -5/3 is extraneous, so x=3.

3.) If $\log_y(b^3) = 1$, what is $\log_{b^2}(y) + \log_b(y)$? (Answer: 9/2)

Solution: Since $\log_y(b^3) = 1$, we have $3 \log_y(b) = 1$, so $\log_y(b) = 1/3$. Therefore, $\log_b(y) = 3$. Now $\log_{b^\wedge 2}(y) = 1/(\log_y(b^2)) = 1/(2 \log_y(b)) = 1/(2*1/3) = 1/(2/3) = 3/2$.
So $3 + 3/2 = 9/2$

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Match 3 Round 6
Discrete Math: Matrices

1.) x= 41 y= -20.5 z= 3
2.) 4
3.) 3, -4

1 point: Give the values of x, y, and z that make the following true:

The inverse of $\begin{bmatrix} 6 & x \\ 2 & 14 \end{bmatrix}$ is $\begin{bmatrix} 7 & y \\ -1 & z \end{bmatrix}$

Solution: In order for the entry in first row of the first column of the inverse to be 7 when the entry in the second row of the second column of the original matrix is 14, the determinant of the original matrix must be 2, so 6*14-2x=2, so x=41.5 Switch the 6 in the original matrix down to z in the inverse by dividing by 2, so z=3. Then y is ½ times the opposite of x, so y=-20.5

Answer: x=41, y=-20.5 or -41/2 , z= 3

2 points: If A is the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B is the matrix $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

give the determinant of the matrix represented by $AB^{=1} + BA^{=1}$.

Solution: $B^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1/2 & 3/2 \\ 1 & -2 \end{bmatrix}$ and $A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

$AB^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1/2 & 3/2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3/2 & -5/2 \\ 5/2 & -7/2 \end{bmatrix}$ and $BA^{-1} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} =$ $\begin{bmatrix} -7/2 & 5/2 \\ -5/2 & 3/2 \end{bmatrix}$ and their sum is $\begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$ so the determinant is 4.

3 points: Find all possible values of x such that the matrix

$\begin{bmatrix} 1 & 2 & x \\ 4 & 5 & 6 \\ 7 & 8 & x^2 \end{bmatrix}$ does not have an inverse. (Answer: 3, -4)

Say we expand by the top row. Then

$1\begin{vmatrix} 5 & 6 \\ 8 & x^2 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & x^2 \end{vmatrix} + x\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} =$

0, so $5x^2 - 48 - 2(4x^2-42) + x(32-35) = 0$, so $-3x^2 - 3x + 36=0$, so $x^2+x-12=0$, so $(x+4)(x-3)=0$, so x=3 or x=-4

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Match 3 Team Round Solutions

1.) __________1.11_____________ 4.) ______4, 16_________________

2.) ________$99.98_______________ 5.) __________4_____________

3.) _________ $\sqrt[3]{130}$ ______________ 6.) _(54-27√3)/2 alternatively 27 – (27√3)/2

$\sqrt[3]{130}$

1.) Express the number $\frac{(6*10^8)(7*10^3)}{(2*10^3)^3(3*10^2)}$ as a number in base 2. Your answer will have a "binary point" instead of a "decimal point".

Solution: This is $42x10^{11}$ / $(24*10^{11})$ = 1.75 = 1 + ½ + ¼ , so the answer is 1.11

2.) Rufus walked into the bank with a check to cash, and the teller mistakenly gave him exactly the same number of cents as there were dollars marked on the check, and exactly half the number of dollars as there were cents marked on the check. When Rufus got home and checked his money, he found that the value of his money was exactly half of what it should have been. Assuming the number of dollars and number of cents are both two-digit numbers, what was the original amount of the check?

Solution: Let d=# of dollars in the original check, and c=# of cents. So the value of the original check was 100d+c. The cashier gave (1/2c) dollars and d cents.. Since this must be half the original amount, we have (1/2c(100)+d) =(1/2)(100d+c), so 50c+d = 50d + 1/2c, so 49.5c = 49d, or 99c = 98d. The only combination of two-digit numbers that works is c=98 and d=99, so the answer is $99.98.

3.) Solve for all possible values of x:

$\log_2(x-4) + \log_{\sqrt{2}}(x^3-2) + \log_{0.5}(x-4) = 14$ (Answer: $x=\sqrt[3]{13}$)

Solution: $\log_{0.5}(x-4) = -\log_2(x-4)$, so those will cancel out, leaving $\log_{\sqrt{2}}(x^3 -2) = 14$. So $(\sqrt{2})^{14} = x^3 – 2$, so $x^3 - 2 = 128$, so $x = \sqrt[3]{130}$.

4.) Find all values of x such that the determinant of $\begin{vmatrix} 1 & 1 & 1 \\ 2 & \log_4(x) & 1 \\ 1 & 2 & \log_4(x) \end{vmatrix}$

is equal to 1

Solution: Expand by top row to get $\begin{vmatrix} \log_4(x) & 1 \\ 2 & \log_4(x) \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & \log_4(x) \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & \log_4(x) \end{vmatrix} = 1$

So $(\log_4 x)^2 - 2 - (2(\log_4 x)-1) + 4 - (\log_4 x) = 1$

So $(\log_4 x)^2 - 3\log_4 x + 2 = 0$, so $(\log_4 x-2)(\log_4 x-1) = 0$, so $\log_4 x=2$ or $\log_4 x=1$, so x=4 or x=16.

5.) f(x) has form $f(x) = ab^x + c$. The graph of 2*f(x-1) + 3 passes through the points (1,3), (2, 7) and (3, 19). Find $f^{-1}(80)$. Find where 3 points f(x) passes through first. Subtract 3 from each y-coordinate to get (1,0), (2,4), and (3,16). Then since the graph was translated right to get f(x-1), translate one unit left to get (0,0), (1,4), and (2,16), and divide by 2 to get (0,0), (1,2), and (2,8).

So we have a+c=0, ab+c=2, and $ab^2+c=8$; a=-c, so ab-a=2, and $ab^2-a = 8$,

So a(b-1)=2, and $a(b^2-1) = 8$. Divide second equation by the first to get b+1=4, so b=3. We know $a*3^0 + c = 0$ and $a*3^1 + c = 2$, so a+c=0 and 3a+c = 2 yields a=1 and c=-1, so the original function is $f(x) = 3^x - 1$. The value of x such that $80 = 3^x - 1$ is x=4.

6.) A regular hexagon is created by connecting the alternating vertices of a dodecagon. If the area of the hexagon is 27(√3)/2 cm^3. Find the total area inside the dodecagon but outside the hexagon.

Solution: The area of the hexagon is comprised of 6 equilateral triangles whose areas add to (27√3)/2, so each triangle must have area (27√3)/12. If $s^2\sqrt{3}/4 = 27\sqrt{3}/12$, then $s^2 = 108/12 = 9$, so s=3. Each of the 6 triangles of outside the hexagon but inside the dodecagon is an isosceles triangle with base 3 and vertex angle 150 degrees. The distance from the center of the dodecagon to each vertex is also 3, because the interior hexagon is composed of equilateral triangles. A segment from the center of the dodecagon to one of its vertices that was not used in forming the hexagon has a length which is the sum of the apothem of the hexagon plus the altitude of one of the isosceles triangles. The apothem of the hexagon by 30-60-90 triangles is (3√3)/2, so the altitude of one of the isosceles triangles is 3 – 3√3/2, or (6-3√3)/2. Since there are 6 triangles, multiply 6 by ½(base)(height) to get 6* (1/2)(3)(6-3√3)/2 = 6* (18-9√3)/4 = 3(18-9√3)/2 = (54-27√3)/2 or 27 – (27√3)/2