FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 1 Round 1
Arithmetic: Percents

1.) $16\frac{2}{3}$ or 50/3

2.) $280 billion

3.) $\frac{1309}{1311}$

1.) At PC Jenney's, an item's original price is discounted by 20% the first week, an additional 25% the second week, and then x% on the third week. If the new price is exactly half the original price, find x.

Solution: Let N = original price. Then 0.5N = (1-x)(0.75)(0.8)N, so
0.5 = (1-x)(0.75)(0.8) and 1-x = (1/2)/((3/4)*(4/5)) = ½*(5/3) = 5/6, so x=1/6. Since we want x as a percent, answer 16 2/3 %

2.) In 2011, the defense budget of a certain country was 40% of the national budget. In 2012, the national budget increased by $80 billion and the defense budget increased by $4 billion and was only 30% of the national budget. What was the country's national budget in 2012?

Solution: Let N = total budget in billions of dollars in 2011 D = defense budget in billions
D = 0.4N, and D+4 = 0.3(N+80), so
0.4N + 4 = 0.3N + 24, so 0.1N = 20, so N = 200. Since the question asks for national budget in 2012, add 80 to get 280.

3.) Stocks ABC and XYZ have the same value on Day 0. On Day 1, the value of stock ABC increases by 5% and the value of stock XYZ decreases by 5%. On Day 2, the value of stock ABC decreases by 15% and the value of stock XYZ increases by 15%. On Day 3, the value of stock ABC increases by 10% and the value of stock XYZ decreases by 10%. Express the ratio of the new value of stock ABC to the new value of stock XYZ as a ratio of integers in lowest terms.

Solution: If the original value of each stock is x, the ratio after three days is

$$\frac{(1.1)(0.85)(1.05)x}{(0.9)(1.15)(0.95)x} = \frac{11*85*105}{9*115*95} = \frac{11*17*21}{9*23*19} = \frac{11*17*7}{3*23*19} = \frac{1309}{1311}$$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 1 Round 2
Algebra I: Equations

1.) x = 13

2.) y = 6 or y = -5

3.) z=-1 or z= -10/9

1.) Solve for x: $3(x+2)-\frac{x-4}{3}=5(x-4)-\frac{x+2}{5}$

Solution: $15[\,3(x+2)-\frac{x-4}{3}=5(x-4)-\frac{x+2}{5}]$ becomes
45(x+2) – 5(x-4) = 75(x-4) – 3(x+2), so 48(x+2) = 80(x-4), so
48x + 96 = 80x – 320, so 32x=416, so x= 13

2.) Solve for y: $3-\frac{y}{y+2}=\frac{36}{y^2-2y-8}$

Solution: $(y^2-2y-8)\,(3-\frac{y}{y+2}=\frac{36}{y^2-2y-8})$ becomes
$3(y^2-2y-8)-y(y-4)=36$, so
$3y^2-6y-24-y^2+4y=36$, so
$2y^2-2y-60=0$, so $2(y-6)(y+5)=0$, so y=6 or y=-5

3.) Solve for z: (6z+7)(8z+9) = (10z+11)(12z+13)
Solution: $48z^2+54z+56z+63=120z^2+130z+132z+143$
So $48z^2+110z+63=120z^2+262z+143$,
So $72z^2+152z+80=0$, so $9z^2+19z+10=0$, so $(9z+10)(z+1)=0$, so
z=-1 or z= -10/9

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 1 Round 3
Geometry: Triangles
And Quadrilaterals

1.) ______ $32+4\sqrt{13}$ ______

2.) ______ $\frac{1200}{49}$ ______

3.) ______ $\frac{15+5\sqrt{3}}{3}$ ______

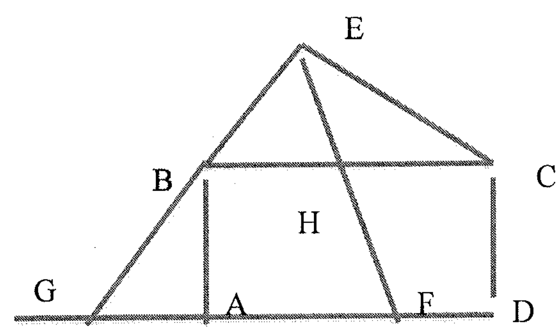
1) An isosceles trapezoid has bases 12 and 20 units and area 96 square units. Find the perimeter of the trapezoid.

Solution: If h= height of trapezoid, then (1/2)h(12+20) = 96, so h=6. The trapezoid is therefore composed a 6 by 12 rectangle, and two triangles with base 4 and height 6. By Pythagorean theorem, the hypotenuse of each triangle is $2\sqrt{13}$, so the perimeter of the trapezoid is $12 + 20 + 2\sqrt{13} + 2\sqrt{13} = 32 + 4\sqrt{13}$

2.) Rectangle ABCD has AB=CD=6 units and AD=BC =8 units. A ray is drawn from A that bisects angle BAD and intersects diagonal BD at E. Find the product of BE and ED.

Solution: Since ABCD is a rectangle, each diagonal has length 10 units by the Pythagorean theorem. By the angle bisector theorem, ray AE must split BD into an 8:6 ratio, so BE and ED must be 40/7 and 30/7, so the product is 1200/49

3.) Triangle BEC is a right triangle with the right angle at E and BE = 5 and EC = $5\sqrt{3}$. The triangle shares side BC of square ABCD as shown. A median of the triangle is drawn from E to side BC and meets side AD of the square at F. How long is AF?



Solution: Suppose EF meets BC at point H.
Extend BE and AD so that they meet at point G. Since BE=5 and EC = $5\sqrt{3}$, BC=10, and since the area of the rectangle is 50 cm^2, then side AB=5. .
Since EH is a median, BH=5, and since you have a 5-$5\sqrt{3}$-10 triangle, angle EBC is 60 degrees, which makes angle BGA also 60 degrees by corresponding angles. Since we know AB=5, then AG = $5/\sqrt{3}$, and BG=$10/\sqrt{3}$. ΔEBH is similar to ΔEGF, so ΔEGF is also equilateral, so EB + BG = AG + AF. $5 + 10/\sqrt{3} = 5/\sqrt{3} + AF$, so AF = $(5 + 5/\sqrt{3})$, or $5 + (5\sqrt{3})/3$, or $(15+5\sqrt{3})/3$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 1 Round 4
Algebra 2:
Simultaneous Equations

1.) ____x=3 y=-1.5_________

2.) _____a=-9/2 b=8/3______

3.) ___p=-1/2 q=2/3 r=1/5___

1.) Solve for x and y

$2x + 3(y+2) = 5(2x+3y)$
$4x - 2(y+3) = x-4y$

Solution: $2x + 3y + 6 = 10x + 15y$ becomes $8x+12y=6$
$4x - 2y - 6 = x - 4y$ $\quad 3x+2y=6$

so $8x+12y=6$ so $-10x=-30$ and $x=3$. $3*3+2y=6$, so $y=-1.5$
$-18x-12y=-36$

2.) Solve for a and b:

$$\frac{2}{a+4} - \frac{3}{b-3} = 5$$

$$\frac{5}{a+4} - \frac{4}{b-3} = 2$$

Solution: Let $c=a+4$ and $d=b-3$, so we have $2/c - 3/d = 5$ so $8/c-12/d=20$
$5/c - 4/d = 2$ $\quad -15/c+12/d=-6$
so $-7/c = 14$, so $c=-1/2$. $2/c-3/d=5$, so $-4-3/d=5$, so $=-3/d=9$, so $d=-1/3$
If $a+4=-1/2$, $a = -9/2$. If $b-3 = -1/3$, then $b=8/3$

3.) Solve for p, q, and r:

$2p + 3q + 5r = 2$ Solution: eliminate r : $-8p-12q-20r=-8$ and $4p-9q+20r=-4$
$4p - 9q + 20r = -4$ $\quad 4p - 9q + 20r = -4$ $\quad 12p+30q-20r=10$
$6p + 15q - 10r = 5$

so $4p-21q=-12$ and $16p+21q=6$, add these to get $12p=-6$, so $p=-1/2$.
Then $16(-1/2) +21q = 6$, so $21q=14$, so $q=2/3$.
And finally $2(-1/2) + 3(2/3) + 5r = 2$, so $-1 + 2 + 5r = 2$, so $r=1/5$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 1 Round 5
Trig: Right Triangles

1.) $\frac{3\sqrt{13}}{13}$

2.) $200\sqrt{3}$

3.) 11

1.) In right triangle ABC, the right angle is at C and tan B = 2/3. What is sin A ?

Solution: The sides must have ratios AC:BC:AB = 2:3:$\sqrt{13}$.
Sin A = BC/AB = 3/$\sqrt{13}$ = 3$\sqrt{13}$/13

2.) If you stand a certain distance from the base of a tower, the angle of elevation to the top of the tower is 60 degrees. If you move back away from the tower an additional 400 feet, the angle of elevation to the top of the tower is 30 degrees. What is the height of the tower in feet?

Solution: tan ($\angle BAC$) = BC/AC = $\sqrt{3}$
tan ($\angle BDC$) = BC/(AC+400) = $\sqrt{3}/3$
so (AC+400)$\sqrt{3}/3$ = (AC)$\sqrt{3}$, multiply by 3 to get (AC)$\sqrt{3}$ + 400$\sqrt{3}$ =3(AC)$\sqrt{3}$, so 2(AC)$\sqrt{3}$ = 400$\sqrt{3}$, so AC =200. and BC is the height of the tower, so it is 200$\sqrt{3}$ feet tall.

3.) A plane descends at an angle of 22 degrees below the horizontal. It reaches a certain altitude and then begins to ascend at an angle of 31 degrees above the horizontal until it reaches its original height. If the difference in horizontal position between the two times when it was at the original height is 10 kilometers, find the total distance traveled by the plane using the table below to the nearest kilometer.

	Sine	Cosine	Tangent
22 degrees	0.37	0.93	0.40
31 degrees	0.52	0.87	0.60

Part of the reason for the "nearest kilometer" is so that the savvy student will only have to do division of decimals to the nearest tenth and recognize that they can stop there. The other reason is to allow for alternate ways of doing the problem that give slightly different answers because of rounding in the table, but should still come out correct to the nearest km. Let the horizontal distance traveled during the first segment be x km, and then the horizontal distance traveled during the second segment is (10-x) km.
If the vertical displacement is h in each case, we have tan 22 = h/x and tan 31 = h/(10-x), so (10-x) tan 31 = x tan 22, so (10-x)(0.6) = x(0.4), so x =6 km and (10-x) is 4 km. The total distance traveled by the plane is found by adding 6/(cos 22) + 4/(cos 31), which is about 6.5 + 4.5, so the total distance is 11 km.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 1 Round 6
Coordinate Geometry

1.) $2\sqrt{65}$

2.) $(\frac{13}{2},\frac{11}{2})$ or $(\frac{-3}{2},\frac{-5}{2})$

3.) -3

1.) An isosceles trapezoid has vertices at (0,0), (3,4), (7,4), and (10,0). Find the perimeter of the quadrilateral formed by connecting the midpoints of the adjacent segments of the trapezoid.

Solution: The midpoints are (1.5, 2), (5,4), (8.5,2), and (5,0).
Using distance formula on one set of points,
$\sqrt{(5-1.5)^2+(4-2)^2}$ = $\sqrt{}((49/4)+4)$ = $\sqrt{65}/2$, and then observe that all other calculations are the same by symmetry, so the perimeter is $4*(\sqrt{65})/2 = 2\sqrt{65}$

2.) The line segment with endpoints (k,1) and (3,k) has length $\sqrt{(130)}$. Find all possible sets of coordinates for the midpoint of the segment.

Solution: $\sqrt{(3-k)^2+(k-1)^2} = \sqrt{130}$
$(3-k)^2 + (k-1)^2 = 130$
$9 - 6k + k^2 + k^2 - 2k + 1 = 130$
$2k^2 - 8k - 120 = 0$, so $k^2 - 4k - 60 = 0$, so $(k-10)(k+6) = 0$, so k=10 or k=-6
The midpoint could be (13/2, 11/2) or (-3/2, -5/2)

3.) A circle of radius 5 units is centered at the origin. Find the product of the slopes of the two lines that are both tangent to the circle and pass through (0,10)

Solution: The two points (x,y) that form lines with (0,10) must be such that $x^2 + y^2= 25$ and the slope y/x is the opposite reciprocal of the slope of the line (y-10)/x since the tangent must be perpendicular to the radius, so
y/x = -x/(y-10), which means $x^2 = y(10-y)$. Substitute to get $10y - y^2 + y^2 = 25$, so

y=5/2, so x=$\pm 5\sqrt{3}/2$. The slopes of the desired lines are $\frac{10-\frac{5}{2}}{\frac{5\sqrt{3}}{2}}$ and $\frac{10-\frac{5}{2}}{\frac{5\sqrt{3}}{2}}$ or $\frac{15}{5\sqrt{3}}$

and $\frac{-15}{5\sqrt{3}}$, which multiplies to -3.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013
Match 1 Team Round Solutions

1.) ______$2500______________ 4.) ______________________

2.) ________ $\frac{-70}{27}$ ________ 5.) ______________________

3.) ________ $\frac{3\sqrt{3}}{4}$ ________ 6.) ______________________

1.) 1.)Three investments totaling $12000 are invested into stocks A, B, and C, which return 4%, 5%, and 6% of the investment per year respectively. The total interest earned at the end of the first year is $610. The interest earned on stock B is $10 less than the sum of the combined interest earned on stocks A and C. How much money was invested in stock A?

Solution: A+B+C=12000

0.04A + 0.05B + 0.06C = 610

0.05B = (0.04A+0.06C) – 10

Substituting the third equation into the second equation for 0.05B gives
0.08A + 0.12C – 10 = 610 or 0.08A + 0.12C = 620 or 8A + 12C = 62000
and multiplying the third equation by 20 and substituting into the first equation for B gives
A + 0.8A + 1.2C – 200 + C = 12000, so 1.8A + 2.2C = 12200
Multiply the first equation by -2.2 and the second equation by 12 gives
-17.6A -26.4C = -136400
21.6A + 26.4C = 146400 so 4A = 10000, and A = $2500

2.) Solve for x: $2[2(2x+3)+3]+3=\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}x+\frac{1}{3}\right)+\frac{1}{3}\right]+\frac{1}{3}$

Solution: The left side multiplies to 8x+21 and the right side multiplies to (1/8)x + (7/12). Set them equal: 8x+ 21 = (1/8)X+(7/12), so that (63/8)x = (-245/12),
So x = (-245*8)/(12*63) = (-245*2)/(3*63) = (-35*2)/(3*9) = -70/27

3.) Two wires of equal length are bent into two shapes, one a square and the other an equilateral triangle. Find the ratio (area of square):(area of triangle). Express your answer in lowest terms.

Solution: Let the length of each wire by x. Then the side of the square is x/4, and the side of the triangle is x/3. The area of the square is $x^2/16$ and the area of the triangle is $(x/3)^2\sqrt{3}/4$ Or $x^2\sqrt{3}/36$. So the ratios are (1/16)/(√3/36) or 36/(16√3) or (36√3)/(48) = (3√3)/4

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013
Match 1 Team Round Solutions

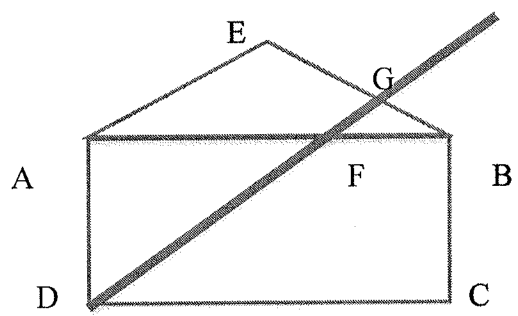
1.) ____________________ 4.) ______________ $\frac{4}{9}$ and $\frac{3}{4}$ __________

2.) ____________________ 5.) $\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}+\sqrt{3}$_ (Same as $\frac{2\sqrt{3}+\sqrt{6}-\sqrt{2}}{2}$)_

3.) ____________________ 6.) __(-3, 6.5) , (-1, 2.5)___

4.) The product of two numbers is 1/3. The sum of their reciprocals is 43/12. Find the two numbers.
xy = 1/3 and 1/x+1/y=43/12. Y=1/(3x), so 1/y = 3x. We have
1/x + 3x = 43/12. Multiply each side by 12x to get $12 + 36x^2 = 43x$
$36x^2 - 43x + 12$ factors to (9x-4)(4x-3) so x=4/9 or 3/4. If x=4/9, then y=3/4, and the reverse, so the numbers are 4/9 and ¾ . $\frac{4}{9}, \frac{3}{4}$

5.) Rectangle ABCD has AB=CD=4 and BC=AD=3. An isosceles triangle ABE with vertex angle of 120 degrees at E is created using AB as the base. [A line is drawn from D intersecting AB at point F and BE at point G such that the measure of angle FDC is 45 degrees. Give the exact perimeter of triangle BFG.



Solution: Since $\angle ADF$ is a bisector, it would be the diagonal of the square with 3 corners A,D, and F, so since AD=3, then AF=3, and BF=4-3 = 1. Now ΔBFG is a triangle with $\angle B = 30$ degrees, and $\angle F = 45$ degrees by corresponding angles. Consider the altitude from G meeting BF at H. Call GH = x, so then BH = x√3, so FH = 1-x√3. Since tan($\angle$GFB) = GH/FH = 1, so x=1-x√3, so x=1/(1+√3). Sin($\angle$GFB) = √2/2 = 1/[(1+√3)GF,] so GF = (√6-√2)/2. Sin ($\angle$GBF) = ½ = so multiply GH by 2 to get BG, so BG = 2/(1+√3) = 2(1-√3)/(-2), OR -1+√3. Adding up all the sides, you get

1 + (√6/2) + (-√2/2) + -1 + √3 = $\frac{2\sqrt{3}+\sqrt{6}-\sqrt{2}}{2}$

6.) Find the coordinates of all points along the perpendicular bisector of the segment whose endpoints are (-3,4) and (-1,5) such that the distance between the desired points and each of the endpoints of the original segment is 2.5 units.

Solution: The slope of the line between the 2 desired points is ½, so the slope of the perpendicular bisector is -2, and it passes through (-2, 4.5), so $y-4.5 = -2(x+2)$, so The two points lie on $y=-2x+1/2$. Take one of the endpoints, say (-3,4). Then $\sqrt{(x+3)^2+(y-4)^2} = 2.5$, so $(x+3)^2 + (y-4)^2 = 25/4$. Substitute to get $(x+3)^2 + (-2x+1/2-4)^2 = 25/4$, so $x^2 + 6x + 9 + 4x^2 + 14x + 49/4 = 25/4$, and $5x^2 + 20x + 15 = 0$, so $x=-3$ or $x=-1$. So the desired points have the same x-coordinates as the original segment, so all the change in distance of 2.5 must come with the y-coordinate. Checking a sketch shows that if $x=-3$, $y = 4+2.5 = 6.5$. If $x=-1$, $y=5-2.5 = 2.5$. So the points are (-3, 6.5) , (-1, 2.5).