NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2019 - SOLUTIONS

Round 1: Arithmetic and Number Theory

1. $\dfrac{\sqrt{0.0009}}{\sqrt{0.000036}}=\dfrac{\sqrt{900}}{\sqrt{36}}=\dfrac{30}{6}=\underline{5}$.

2. The least common multiple of 9, 12, 15, and 17 is $2^2 \cdot 3^2 \cdot 5 \cdot 17 = 3060$. So the product must be divisible by 3060.
Case 1: Two-digit numbers divisible by 17
The largest such number is $5 \cdot 17 = 85$.
The three-digit number must be of the form $2^2 \cdot 3^2 \cdot k = 36k$.
The largest such number with a hundreds digit of 3 occurs when $k = 11$, resulting in the product of (85)(396) = 33,660.
Case 2: Numbers in the 300's which are divisible by 17
The 6 possibilities are $17 \cdot 18 = 306$, 323, 340, 357, 374, $17 \cdot 23 = 391$.
If any of these are to be considered, they must produce a product greater than 33660.
Since $374 \cdot 89 = 33286$, the only possible product is $391 \cdot (8\_)$.
391 accounts for divisibility by 17, $(8\_)$ must account for divisibility by 9, 12, and 15.
Since the $\text{LCM}(9,12,15) = 2^2 \cdot 3^2 \cdot 5 = 180$, there is no possible two-digit multiplier $(8\_)$.
Therefore, the answer is **<u>33,660</u>**.

3. We have $\dfrac{10x+y}{99}+\dfrac{10y+x}{99}=\dfrac{z}{9} \Rightarrow 11x+11y=11z \Rightarrow x+y=z$.

If $z = 9$, we have (x,y) = (8,1), (7, 2), (6, 3), (5, 4) or vice versa, giving 8 possible triples.
We note that the number of triples for each value of z will be $z-1$ for odd digits, and $z-2$ for even digits, since ordered pairs where $x = y$ must be excluded.
Thus, there will be 8 + 6 + 6 + 4 + 4 + 2 + 2 + 0 = <u>32</u> triples.

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Round 2 Algebra 1

1. Let x = the weight of the whole ham and y = the weight of the half ham. Then the combined weight is $x+y$, the combined price is $2x+3y$, and the average price per pound is $\dfrac{2x+3y}{x+y}$. Then $\dfrac{2x+3y}{x+y}=\dfrac{9}{4}\Rightarrow 8x+12y=9x+9y\Rightarrow 3y=x$. Thus, $x:y=\underline{\mathbf{3:1}}$.

2. $x>0>y\Rightarrow\dfrac{|x-y|}{x+2|y|}=\dfrac{x-y}{x-2y}=\dfrac{3}{4}$

   $\Rightarrow 4x-4y=3x-6y$

   $\Rightarrow x=-2y$

   Therefore, $\dfrac{x}{y}=\dfrac{-2y}{y}=\underline{\mathbf{-2}}$.

3. If the mode is 37, there must be at least two 37's. Since the median is 58, there must be exactly two 37's. The remaining two integers must be greater than 58 and unequal (again because 37 is <u>the</u> mode). To minimize the average, we take the two unknown integers to be 59 and 60. This yields $A=\dfrac{2(37)+58+59+60}{5}=\dfrac{251}{5}=50.2\Rightarrow\lceil A\rceil=\underline{\mathbf{51}}$.

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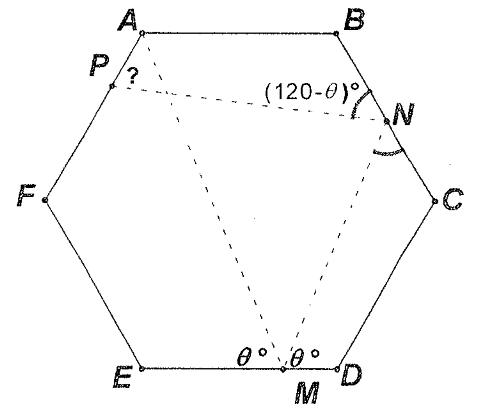
Round 3 Geometry



1. The general case is more instructive.

   If $m\angle AME = \theta°$, then $m\angle NMD = \theta°$, making

   $m\angle MNC = 360° - (120+120+\theta)° = (120-\theta)°$.

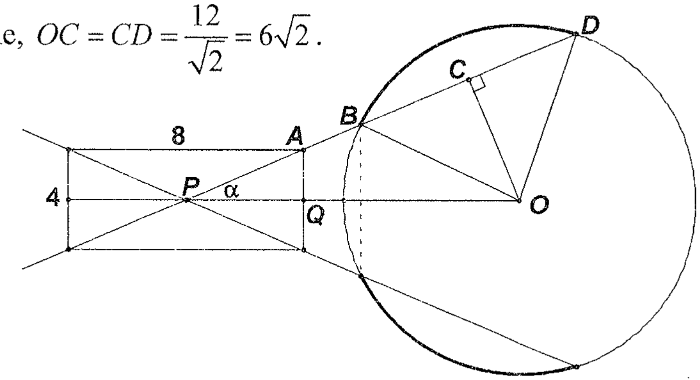
   Then $m\angle BNP = (120-\theta)°$, making

   $m\angle NPA = 360° - (120-\theta+120+120)° = \theta°$.

   So $m\angle NPA = \underline{\mathbf{70°}}$.

2. Since ΔBOD is a 45-45-90 right triangle, $OC = CD = \frac{12}{\sqrt{2}} = 6\sqrt{2}$.

   Since ΔAQP is a *right* triangle (shown below) and shares a common angle (α) with ΔOCP, $\Delta AQP \sim \Delta OCP$.



   Therefore, $\frac{AQ}{OC} = \frac{PQ}{PC}$.

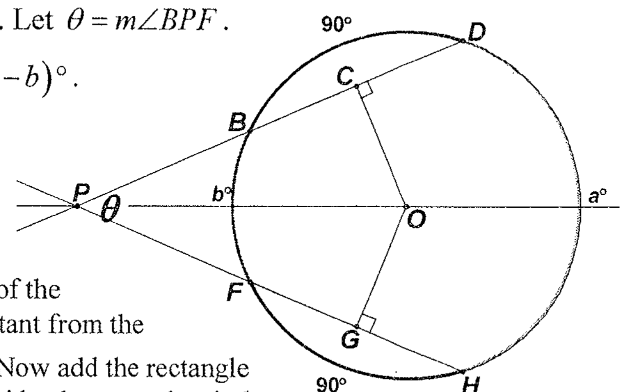
   Switching the means, $\frac{AQ}{PQ} = \frac{OC}{PC}$.

   But, from the dimensions of the rectangle, we have $\frac{AQ}{PQ} = \frac{2}{4} = \frac{1}{2}$.

   Cross multiplying $\frac{OC}{PC} = \frac{1}{2} \Rightarrow PC = 2(OC) = 12\sqrt{2} \Rightarrow PD = \underline{\mathbf{18\sqrt{2}}}$.

   <u>**FYI**</u>:

   Besides showing that ΔAQP is a *right* triangle, we should show that the rectangle could not be drawn with the longer side closest to the circle. Let $\theta = m\angle BPF$.



   Since $\angle BPF$ is formed by two secant lines, $\theta = \frac{1}{2}(a-b)°$.

   We also know that $a+b = 180° \Rightarrow a < 180°$.

   $\theta = \frac{1}{2}(a-(180-a))° = (a-90)°$

   $\Rightarrow \theta < 90° \Rightarrow \angle BPF$ is acute, implying the side of the rectangle opposite $\angle BPF$ must be a shorter side of the rectangle. Additionally, since the 90° arcs are equidistant from the center of the circle, the ray $\overrightarrow{PO}$ is an angle bisector. Now add the rectangle with diagonals lying on these secant lines and a short side closest to the circle.

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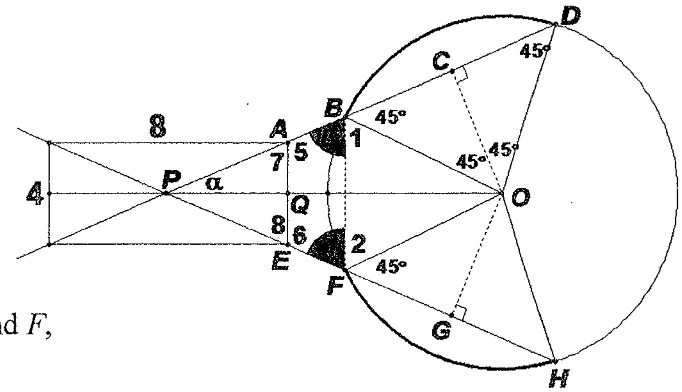
Round 3 Geometry

2. continued



$PH = PD = 18\sqrt{2}$.

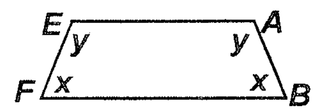
$\left.\begin{array}{l} PB = PD - BD \\ PF = PH - FH \end{array}\right\} \Rightarrow$

$PB = PF \Rightarrow m\angle 3 = m\angle 4$

Since both P and O are equidistant from B and F,

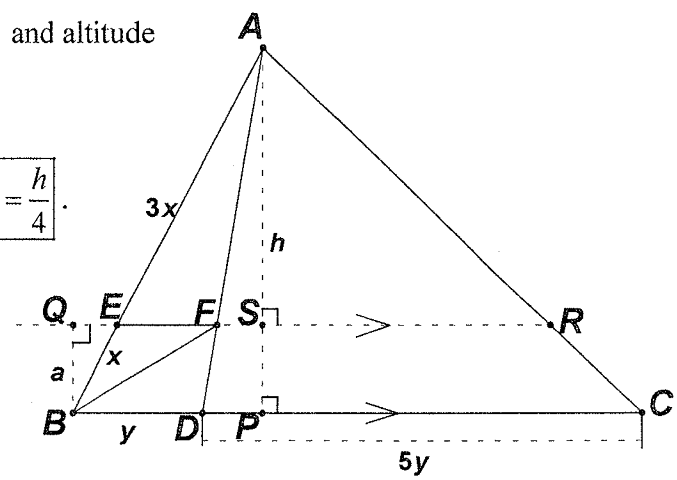
$\overline{OP}$ is the perpendicular bisector of $\overline{BF}$

$PA = PE \Rightarrow m\angle 7 = m\angle 8 \Rightarrow m\angle 5 = m\angle 6$



From the diagram at the right, we see that $2x + 2y = 360° \Rightarrow x + y = 180° \Rightarrow \overline{AE} \parallel \overline{BF}$. Thus, $\overline{OP}$ is also the perpendicular bisector of $\overline{AE}$, and ΔAQP must be a right triangle.

3. Draw line $\overleftrightarrow{EF}$, altitude $\overline{AP}$ in ΔABC $(h = AP)$ and altitude $\overline{BQ}$ in ΔBEF $(a = BQ)$. $\overline{QR} \parallel \overline{BC} \Rightarrow PF = a$.



Since $\Delta BQE \sim \Delta ASE$, $\dfrac{a}{h-a} = \dfrac{1}{3} \Rightarrow h = 4a \Rightarrow \boxed{a = \dfrac{h}{4}}$.

Since $\Delta AEF \sim \Delta ABD$,

$\dfrac{EF}{BD} = \dfrac{AE}{AB} \Rightarrow \dfrac{EF}{y} = \dfrac{3}{4} \Rightarrow EF = \dfrac{3}{4}y$.

Therefore, the area of

$\Delta EFB = \dfrac{1}{2} BQ \cdot EF = \dfrac{1}{2} \cdot \dfrac{h}{4} \cdot \dfrac{3}{4} y = 4 \Rightarrow 3hy = 128$.

Finally, the area of $\Delta ABC = \dfrac{1}{2} h \cdot 6y = 3hy = \mathbf{\underline{128}}$.

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Round 4 – Algebra 2

1. $9^{x+5/2} - 27^{2y-8} = 3^{2x+5} - 3^{6y-24} = 2$. Consider *integer* powers of 3.
$\left\{ \ldots, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, \underline{\underline{3^0 = 1, 3^1 = 3}}, 9, 27, \ldots \right\}$ The *only* way this difference can equal 2 for integer values of x and y, is if $3^{2x+5} = 3$ and $3^{6y-24} = 1$. $(2x+5, 6y-24) = (1,0) \Rightarrow (x,y) = \underline{(-2,4)}$.
Why are we sure this was the only way to achieve a difference of 2? Let $b < a$ be integers.
Case 1: If $a \le 0$, then $0 < 3^b < 3^a \le 1 \Rightarrow 3^a - 3^b < 1$.
Case 2: If $a \ge 1$, then $3^a - 3^b = 3^a\left(1 - 3^{b-a}\right) \ge 3^1\left(1 - 3^{-1}\right) = 2$
Thus, for the left-hand side to be exactly 2, we must have $a = 1$ and $b = 0$.

2. Letting $y = \sqrt[3]{x}$, we have $y + \frac{1}{y} = 3$, which is quickly transformed to the quadratic equation $y^2 - 3y + 1 = 0$. Applying the quadratic formula, $y = \sqrt[3]{x} = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$.
Cubing both sides gives $x = \left(\frac{3 \pm \sqrt{5}}{2}\right)^3 = \frac{27 \pm 27\sqrt{5} + 45 \pm 5\sqrt{5}}{8} = \frac{72 \pm 32\sqrt{5}}{8} = \underline{9 \pm 4\sqrt{5}}$.

3. Note that for $x = 2$, $f(2) - 2f(-1) = 4$ and if $x = -1$, $f(-1) - 2f(2) = 1$.
Letting $f(2) = x$ and $f(-1) = y$, we have $x - 2y = 4$ and $y - 2x = 1$.
Solving, we obtain $y = -3$ and $x = -2$ and $f(2) = \underline{-2}$.

Alternate solution (Finding an *explicit* formula for the *implicit* $f(x) - 2f(1-x) = x^2$):

Let $x = 1 - y$. Then: $f(1-y) - 2f(1-(1-y)) = (1-y)^2 \Leftrightarrow f(1-y) - 2f(y) = 1 - 2y + y^2$,

for all real y. This can be rewritten as $f(1-x) = 2f(x) + 1 - 2x + x^2$, for all real x.

Substituting in the given equation, we have $f(x) - 2\left(2f(x) + 1 - 2x + x^2\right) = x^2$

$\Leftrightarrow f(x) - 4f(x) - 2\left(1 - 2x + x^2\right) = x^2 \Rightarrow \boxed{f(x) = -x^2 + \frac{4}{3}x - \frac{2}{3}}$. Now we verify that this

$f(x)$ satisfies the stated conditions for all real x. $f(x) - 2f(1-x) =$

$$\left(-x^2 + \frac{4}{3}x - \frac{2}{3}\right) - 2\left(-(1-x)^2 + \frac{4}{3}(1-x) - \frac{2}{3}\right) = -x^2 + \frac{4}{3}x - \frac{2}{3} + 2 - 4x + 2x^2 - \frac{8}{3} + \frac{8}{3}x + \frac{4}{3} = x^2$$

Finally, we compute $f(2) = -4 + \frac{8}{3} - \frac{2}{3} = -4 + 2 = \underline{-2}$.

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Round 4 - Algebra 2

The following is a chart of function values of the given implicit function $f(x)-2f(1-x)=x^2$ for consecutive integer x-values, differences between those function values, and differences between those differences (referred to as second differences)

x	$f(x)$	**1st Diff**	**2nd Diff**
-2	-22/3		
-1	-3	-3 – (-22/3) = 13/3	
0	-2/3	7/3	7/3 – (13/3) = -2
1	-1/3	1/3	-2
2	-2	-5/3	-2
3	-17/3	-11/3	-2
4	-34/3	-17/3	-2

Here's an important principle about these differences:
If second differences between function values are identical for all consecutive x-values, then the function must be a polynomial (for integer values of x) of not higher than second degree.
Similar statements can be made for identical n^{th} differences.

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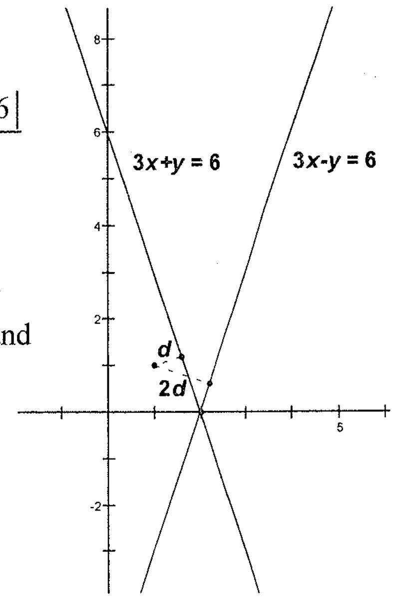
Round 5 – Analytic Geometry

1. $(a+2)(2a+1)=-1 \Rightarrow 2a^2+5a+3=0 \Rightarrow (2a+3)(a+1)=0$. Therefore, $a=$ **-1 or $-\frac{3}{2}$**.

2. Using the point-to-line distance formula, $\frac{|3a-b-6|}{\sqrt{10}}=\frac{2|3a+b-6|}{\sqrt{10}}$

$\Rightarrow 3a-b-6=6a+2b-12$ or $3a-b-6=-(6a+2b-12)$.

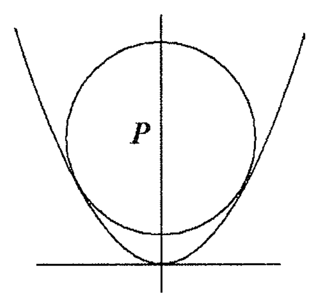
The first simplifies to $\boldsymbol{a+b=2}$, and the only possible solution in positive integers is **(1, 1)**. The second simplifies to $\boldsymbol{9a+b=18}$, and the only solution in positive integers is **(1, 9)**.



3. Let the radius of the circle be r. The circle's equation is $x^2+\left(y-\frac{25}{2}\right)^2=r^2$ and we want its intersection with $y=x^2$. Substituting, we have $y+y^2-25y+\frac{625}{4}=r^2$.

$\Rightarrow y^2-24y+\left(\frac{625}{4}-r^2\right)=0$. For there to be one solution, the discriminant must be 0.

Set $24^2-4\left(\frac{625}{4}-r^2\right)=0 \Rightarrow 4r^2=49$, so $r=$ **$\frac{7}{2}$**.



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Round 6 – Trig and Complex Numbers

1. Let $\theta = \text{Cos}^{-1} x$, for $0 \le \theta \le \pi$. Then $\cos\theta = x$, which implies that $\sin\theta = \sqrt{1-x^2}$, since the θ must be quadrants 1 and 2. Substituting gives $\sqrt{1-x^2} = x \Rightarrow 1-x^2 = x^2 \Rightarrow 2x^2 = 1$ $\Rightarrow x = \pm\frac{\sqrt{2}}{2}$. However, $x = -\frac{\sqrt{2}}{2}$ is extraneous, and $x = \underline{\mathbf{\frac{\sqrt{2}}{2}}}$ only.

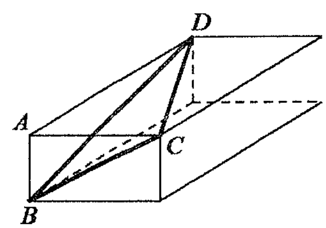
Alternately, we require that θ be an angle between 0 and π, i.e., in quadrant 1 or 2, for which sine and cosine are equal.

The only possibility is $45°$ (or $\frac{\pi}{4}$ radians), producing $x = \underline{\mathbf{\frac{\sqrt{2}}{2}}}$.

2. $BC^2 = 8^2 + 15^2 = 289 \Rightarrow BC = 17$.

$BD^2 = 8^2 + 20^2 = 464 = 16(29) \Rightarrow BD = 4\sqrt{29}$.

$CD^2 = 15^2 + 20^2 = 625 \Rightarrow CD = 25$



Using the Law of Cosines on ΔBCD, we have

$464 = 289 + 625 - 2\cdot17\cdot25\cdot\cos(\angle DCB)$.

$\cos(\angle DCB) = \frac{289+625-464}{2\cdot17\cdot25} = \frac{914-464}{2\cdot17\cdot25} = \frac{\cancel{450}^{9}}{\cancel{2}\cdot17\cdot\cancel{25}}$.

Thus, $(m,n) = \underline{\mathbf{(9,17)}}$.

3. $\tan\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2} = \tan 45°\cdot\tan^2 22.5° = \tan^2 22.5°$. From $1+\tan^2\theta = \sec^2\theta$, we have $\tan^2 22.5° = \sec^2 22.5° - 1$. From $\left|\cos\frac{\theta}{2}\right| = \sqrt{\frac{1-\cos\theta}{2}}$, we have

$\cos 22.5° = \sqrt{\frac{1+\cos 45°}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}}$, which gives us $\sec^2 22.5° = \frac{4}{2+\sqrt{2}} = 4-2\sqrt{2}$.

Therefore, $\tan\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2} = \tan^2 22.5° = \sec^2 22.5° - 1 = \underline{\mathbf{3-2\sqrt{2}}}$.

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Team Round

1. There are $\frac{9!}{3!\cdot 2!}$ distinct permutations of MASSASOIT. If we remove the three S's, there are $\frac{6!}{2!}$ distinct permutations of the *remaining* letters. If we group the three S's together as single unit, there are 7 distinct places in any permutation of the other six letters in which to put the group of S's, giving $7\cdot\frac{6!}{2!}$ arrangements in which the S's are together.

The probability that the S's will *not* be together is

$$\frac{\frac{9!}{3!\cdot 2!}-7\cdot\frac{6!}{2!}}{\frac{9!}{3!\cdot 2!}}=\frac{9!-7!\cdot 3!}{9!}=1-\frac{6}{9\cdot 8}=1-\frac{1}{12}=\frac{11}{12}.$$ Therefore, $a+b=\underline{\mathbf{23}}$.

2. Adding and multiplying gives $\left(\frac{ab+2}{b}\right)\left(\frac{bc+2}{c}\right)\left(\frac{ac+2}{a}\right)=3\cdot 4\cdot 5=60$. The left side becomes $\frac{a^2b^2c^2+2abc^2+2a^2bc+2ab^2c+4ac+4bc+4ab+8}{abc}$. This simplifies first to

$\frac{(abc)^2+2c(abc)+2a(abc)+2b(abc)+4(ac+bc+ab)+8}{abc}$, and then to

$abc+\frac{8}{abc}+2c+2a+2b+\frac{4}{b}+\frac{4}{a}+\frac{4}{c}$, which equals

$abc+\frac{8}{abc}+2\left(c+\frac{2}{a}\right)+2\left(a+\frac{2}{b}\right)+2\left(b+\frac{2}{c}\right)=abc+\frac{8}{abc}+2\cdot 5+2\cdot 3+2\cdot 4=60$.

Thus, $abc+\frac{8}{abc}=60-24=\underline{\mathbf{36}}$.

Alternately, $\left(a+\frac{2}{b}\right)\left(b+\frac{2}{c}\right)\left(c+\frac{2}{a}\right)=3\cdot 4\cdot 5=60$

$\Leftrightarrow\left(ab+\frac{2a}{c}+2+\frac{4}{bc}\right)\left(c+\frac{2}{a}\right)=abc+2a+2c+\frac{4}{b}+2b+\frac{4}{c}+\frac{4}{a}+\frac{8}{abc}=60$. Regrouping,

$abc+\frac{8}{abc}+\left(2a+\frac{4}{b}\right)+\left(2b+\frac{4}{c}\right)+\left(2c+\frac{4}{a}\right)=abc+\frac{8}{abc}+2\left[\left(a+\frac{2}{b}\right)+\left(b+\frac{2}{c}\right)+\left(c+\frac{2}{a}\right)\right]=60$

$\Rightarrow abc+\frac{8}{abc}+2(3+4+5)=60\Rightarrow abc+\frac{8}{abc}=60-24=\underline{\mathbf{36}}$.

Team Round

3. $\dfrac{1^2+2^2+3^2+\ldots+(n-1)^2+n^2}{1^3+2^3+3^3+\ldots+(n-1)^3+n^3}=\dfrac{\frac{n(n+1)(2n+1)}{6}}{\left(\frac{n(n+1)}{2}\right)^2}=\dfrac{2(2n+1)}{3n(n+1)}$. Then $\dfrac{4n+2}{3n^2+3n}>\dfrac{1}{15}$.

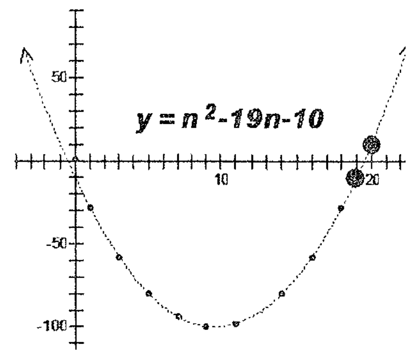
$60n+30>3n^2+3n \Leftrightarrow 3n^2-57n-30>0$

$\Leftrightarrow n^2-19n-10<0 \Leftrightarrow n(n-19)-10<0.$

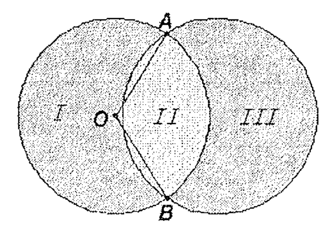
We note that if $n=19$ we have $0-10<0$ which is true.

However, if $n\ge 20$, we have $n(n-19)-10\ge 20(1)-10>0$.

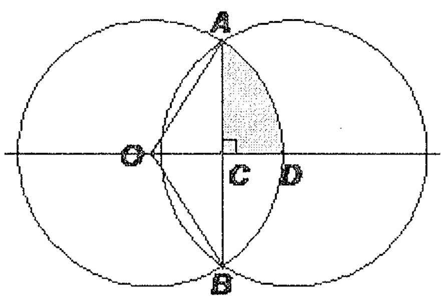
The graph of $y=n^2-19n-10$ is a series of discrete points located on an upward opening parabola with vertex at $n=9.5$, which has strictly increasing function values for $n>9.5$.



Thus, the largest positive integer satisfying the original inequality is <u>19</u>.



4. Let $m\angle AOC=\dfrac{\theta}{2}$. Since the area of region II, the intersection of the two circles, is $\dfrac{1}{3}$ the area of circle O, the area of the region bounded by chord $\overline{AB}$ and minor arc $\widehat{ADB}$ is 1/6 the area of circle O, so the area of the region bounded by $\overline{AC}$, $\overline{CD}$, and minor arc $\widehat{AD}$ is 1/12 the area of circle O. Note that $OC=\cos\dfrac{\theta}{2}$ and $AC=\sin\dfrac{\theta}{2}$.



The area of sector AOD minus the area of ΔACO equals the area of the shaded region ACD. This gives $\dfrac{\theta/2}{2\pi}\cdot\pi\cdot 1^2-\dfrac{1}{2}\cdot\cos\dfrac{\theta}{2}\cdot\sin\dfrac{\theta}{2}=\dfrac{1}{12}\cdot\pi\cdot 1^2$ which simplifies to $\dfrac{\theta}{4}-\dfrac{1}{4}\sin\theta=\dfrac{\pi}{12}$, giving $\sin\theta=\theta-\dfrac{\pi}{3}=\dfrac{3\theta-\pi}{3}$. Therefore, $k+m+n=\underline{7}$. Alternately, the required area is twice the difference between the area of minor sector AOB and triangle AOB.

$2\left(\dfrac{1}{2}\cdot 1^2\cdot\theta-\dfrac{1}{2}\cdot 1^2\cdot\sin\theta\right)=\theta-\sin\theta \Rightarrow \dfrac{\pi-(\theta-\sin\theta)}{\theta-\sin\theta}=\dfrac{2}{1}\Rightarrow \pi-\theta+\sin\theta=2\theta-2\sin\theta\Rightarrow$

$3\sin\theta=3\theta-\pi\Rightarrow\sin\theta=\dfrac{3\theta-\pi}{3}\Rightarrow k+m+n=3+1+3=\underline{7}$.

[**FYI**: $\theta\approx 112.7°$ or $1.97^{radians}$ and $\sin\theta\approx 0.92$.]

Team Round

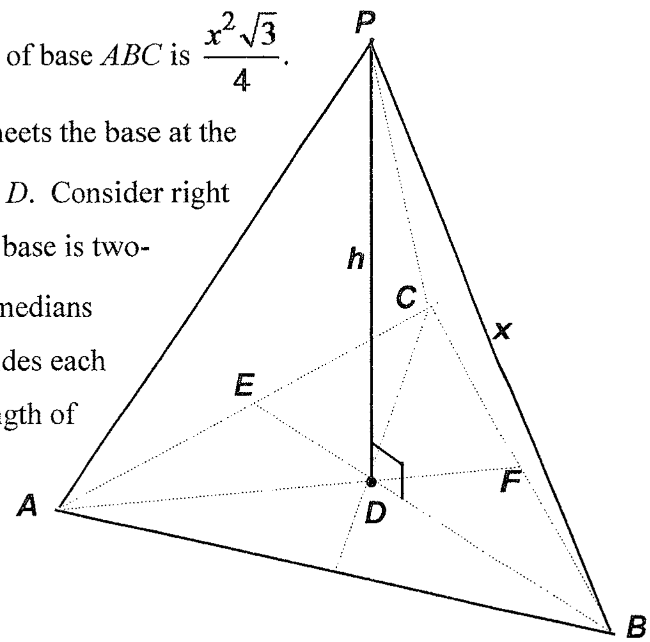
5. Since there are 3 possible values of AB and n possible values of AC, there are $3n$ possible triangles.

   Using $\frac{1}{2}bc\sin 30° = \frac{1}{4}bc$ to calculate areas, we have the average area is

$$\frac{1}{3n}\cdot\frac{1}{4}\left(1\cdot\sum_{k=1}^{k=n}k + 2\cdot\sum_{k=1}^{k=n}k + 3\cdot\sum_{k=1}^{k=n}k\right) = \frac{1}{12n}\left(6\cdot\sum_{k=1}^{k=n}k\right) = \frac{1}{2n}\cdot\frac{n(n+1)}{2} = \frac{n+1}{4} = 20 \Rightarrow n = \underline{79}.$$

6. Let the edges have length x. Then the area of base ABC is $\frac{x^2\sqrt{3}}{4}$.

   An altitude from the vertex P to the base meets the base at the intersection of the medians $\overline{AF}$ and $\overline{BE}$ at D. Consider right triangle PDB whose height is h and whose base is two-thirds the length of median $\overline{BE}$, since the medians intersect at a point of concurrency that divides each median into lengths in a 2 : 1 ratio. The length of the median is $\frac{x\sqrt{3}}{2}$, which gives



$$DB = \frac{2}{3}\cdot\frac{x\sqrt{3}}{2} = \frac{x}{\sqrt{3}}.$$ Thus,

$$h^2 = x^2 - \left(\frac{x}{\sqrt{3}}\right)^2 = \frac{2}{3}x^2 \Rightarrow h = \frac{x\sqrt{2}}{\sqrt{3}}$$ and the volume is

$$\frac{1}{3}Bh = \frac{1}{3}\cdot\frac{x^2\sqrt{3}}{4}\cdot\frac{x\sqrt{2}}{\sqrt{3}} = \frac{x^3\sqrt{2}}{12} = 1.$$

Since $h^3 = \frac{x^3\cdot 2\sqrt{2}}{3\sqrt{3}}$ and $x^3 = \frac{12}{\sqrt{2}}$, $h^3 = \frac{12}{\sqrt{2}}\cdot\frac{2\sqrt{2}}{3\sqrt{3}} = \frac{8}{\sqrt{3}} = \underline{\frac{8\sqrt{3}}{3}}$.