MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2015 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. We require that $2^A + A^2 = 100$. Since the sum is even, A must be even.
   By brute force, $2^6 + 6^2 = 64 + 36 = 100$, yielding $A = \underline{\mathbf{6}}$

2. If a = 1 to 5, b = 1 to 4, giving 20 ordered pairs. If a = 6 to 8, b = 1 to 3, giving 9 ordered pairs. If a = 9, b = 1 or 2, giving 2 ordered pairs. Thus, there are a total of **<u>31</u>** ordered pairs.

3. The number of ways of filling the grid to satisfy the stated property is

4	3	2
2	1	1
4	3	2

, whereas a random fill could be done in

9	8	7
6	5	4
3	2	1

ways. Thus, the probability is

$$\frac{2(4!)^2}{9!} = \frac{\cancel{2\cdot 4}\cdot\cancel{3\cdot 2}}{9\cdot\cancel{8}\cdot 7\cdot\cancel{6}\cdot 5} = \frac{1}{\cancel{315}} \quad \frac{1}{35}$$

Round 2 Algebra 1

1. $312_{(x-1)} = 211_{(x+1)} \Leftrightarrow 3(x-1)^2 + (x-1) + 2 = 2(x+1)^2 + (x+1) + 1$
   $\Leftrightarrow 3x^2 - 5x + 4 = 2x^2 + 5x + 4 \Leftrightarrow x^2 - 10x = x(x-10) = 0 \Rightarrow x = \underline{\mathbf{10}}$.

2. $\left(\frac{1}{a-3}\right)^2 = \frac{25}{144} \Rightarrow a = 3 \pm \frac{12}{5}$ $\boxed{a = \frac{27}{5}, \frac{3}{5}}$

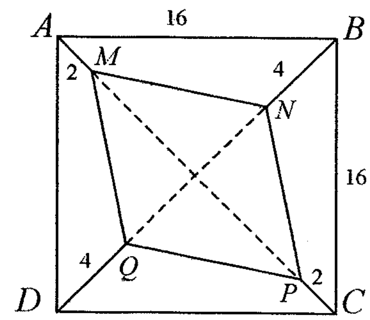
3. 1^{160} and 9^0 both equal 1 and are ignored. Convert each of the in-between values to powers of 20. $2^{140} = (2^7)^{20} = 128^{20}$, $3^{120} = (3^6)^{20} = 729^{20}$, $4^{100} = (4^5)^{20} = 1024^{20}$,
   $5^{80} = (5^4)^{20} = 625^{20}$,
   $6^{60} = (6^3)^{20} = 216^{20}$, $7^{40} = (7^2)^{20} = 49^{20}$
   The larger the base is, the larger the power. Thus, $(A, B) = \underline{\left(\mathbf{4^{100}, 3^{120}}\right)}$.

Round 3 – Geometry

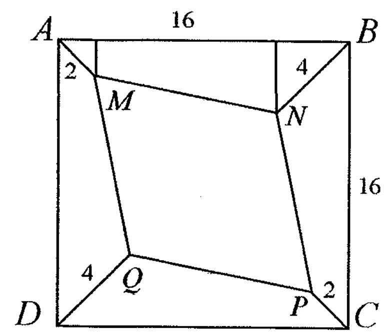
1. Since the exterior angles of any triangle (one per vertex) always total 360°, we have $7n+10n+13n=30n=360 \Rightarrow n=12 \Rightarrow (m\angle 1, m\angle 2, m\angle 3)=(84,120,156)$ $\Rightarrow (m\angle 4, m\angle 5, m\angle 6)=(96,60,24)$. Thus, the required ratio is **8 : 5 : 2**

2. Method 1: Since the 4 trapezoids are congruent, *MNPQ* is a rhombus. Since $AC=DB=16\sqrt{2}$, then $MP=16\sqrt{2}-4$ and $QN=16\sqrt{2}-8$. Using the area formula $\frac{d_1 \cdot d_2}{2}$ gives

$$\frac{(16\sqrt{2}-4)(16\sqrt{2}-8)}{2}=\frac{512-64\sqrt{2}-128\sqrt{2}+32}{2}=\boxed{272-96\sqrt{2}}.$$



Method 2: Drop perpendiculars from *M* and *N* to $\overline{AB}$. The first is $\sqrt{2}$ units long, the second is $2\sqrt{2}$ units long. They divide *AMNB* into two 45-45-90 right triangles with areas of $\frac{1}{2}\cdot\sqrt{2}\cdot\sqrt{2}=1$ and $\frac{1}{2}\cdot 2\sqrt{2}\cdot 2\sqrt{2}=4$, respectively, and one trapezoid whose bases have lengths of $\sqrt{2}$ and $2\sqrt{2}$ and whose height is $16-3\sqrt{2}$.



The area of the trapezoid is $\frac{1}{2}(16-3\sqrt{2})(3\sqrt{2})=24\sqrt{2}-9$. Thus, the area of *AMNB* is $24\sqrt{2}-9+5=24\sqrt{2}-4$. The sum of the areas of the four congruent trapezoids is $96\sqrt{2}-16$. Then the area of *MNPQ* is $16^2-(96\sqrt{2}-16)=\boxed{272-96\sqrt{2}}$.

3. Let h denote the altitude from the vertex of the isosceles triangle to its base. Since the radius R of a circle circumscribed about a triangle with sides x, y, and z is $\frac{xyz}{4\cdot \text{area of } \Delta}$, we have $R=\frac{a^2c}{4\left(\frac{1}{2}hc\right)}=\frac{a^2}{2h}$. From the Pythagorean theorem,

$a^2=h^2+\left(\frac{c}{2}\right)^2 \Rightarrow h^2=\frac{4a^2-c^2}{4}$. Squaring and substituting, $R^2=\boldsymbol{\frac{a^4}{4a^2-c^2}}$.

Alternate Solution:

The diameter d of the circumscribed circle of a triangle is $\frac{\text{side}}{\sin(\text{opposite angle})}$. Consider A as the vertex angle, then $\sin B=\sqrt{\frac{4a^2-c^2}{2a}}$, so $d=\frac{2a^2}{\sqrt{4a^2-c^2}}$.

Round 4 – Algebra 2

1. $n=0\Rightarrow 1+1=\underline{2}$; $n=1\Rightarrow i+\frac{1}{i}=i+(-i)=\underline{0}$; $n=2\Rightarrow i^2+\frac{1}{i^2}=-1+(-1)=\underline{-2}$

$n=3\Rightarrow i^3+\frac{1}{i^3}=-i+(i)=0$ … and the cycle repeats.

2. $\left(\frac{\log 0.\bar{3}}{\log 81}\right)=\log_{81}\left(3^{-1}\right)=-\log_{81}3=-\frac{1}{4},\ \frac{\log 8}{\log 4}=\log_4 8=\frac{3}{2}$

$-\frac{1}{4}x^2+\frac{3}{2}x+3-c=0\Leftrightarrow x^2-6x-4(3-c)=0$

Equal roots require that the discriminant be zero. $36+16(3-c)=0\Rightarrow c=\frac{84}{16}=\underline{\mathbf{\frac{21}{4}}}$

3. The expansion of $\left(1+x^2\right)^t$ is $\binom{t}{0}1+\binom{t}{1}\left(x^2\right)^1+\binom{t}{2}\left(x^2\right)^2+\binom{t}{3}\left(x^2\right)^3+\binom{t}{4}\left(x^2\right)^4+\ldots$

Therefore, $6\cdot\binom{t}{2}=\binom{t}{4}\Rightarrow\frac{6\cancel{t(t-1)}}{2!}=\frac{\cancel{t(t-1)}(t-2)(t-3)}{4!}$

$\Rightarrow 72=(t-2)(t-3)\Rightarrow t^2-5t-66=0\Rightarrow(t-11)(t+6)=0\Rightarrow t=11$

Finally, the ninth term in the expansion is $\binom{11}{8}\left(x^2\right)^8=\frac{11\cdot 10\cdot 9}{1\cdot 2\cdot 3}x^{16}=\underline{\mathbf{165x^{16}}}$.

Round 5 – Analytic Geometry

1. The equation can be rewritten as $(x-4)^2+(y+7)^2=25$, the graph of which is a circle of radius 5. The area of the triangle formed by two radii and a chord is found by $\frac{1}{2}ab\sin\theta$ which gives $\frac{25\sqrt{2}}{4}$. The area of the 45° sector is $\frac{1}{8}\cdot 25\pi$. Subtracting gives $\frac{25}{8}\left(\pi-2\sqrt{2}\right)$. The ordered quadruple is $\underline{\mathbf{(25,8,2,2)}}$.

2. The given equation is that of a parabola with vertex $(-3,4)$. The required x-coordinates are 1 and 13. The vertices of the trapezoid are $(1,2),(1,6),(13,8)$, and $(13,0)$. The area is $\frac{1}{2}(4+8)(13-1)=\underline{72}$.

3. Given ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, with $V(0,-2\sqrt{10})\rightarrow a^2=40$. Substituting $P(-3,-4)$ gives $\frac{9}{b^2}+\frac{2}{5}=1\rightarrow b^2=15$. Now $F(0,f)$ becomes $F(0,\sqrt{40-15})$ or $F(0,5)$. The slope of L_1 is $\frac{0+4}{1+3}=1$. L_1 has equation $x-y=1$. L_2 has slope -1 and passes through $F(0,5)$ so its equation is $x+y=5$. $\therefore$ $\underline{\boldsymbol{Q(3,2)}}$

Round 6 – Trig and Complex Numbers

1. Applying the exponents of 4 and 6, the expression becomes $\left(4cis\frac{7\pi}{6}\right)\left(27cis\frac{5\pi}{3}\right)$.

   Multiplying gives $108cis\frac{17\pi}{6} = \underline{\mathbf{-54\sqrt{3}+54i}}$.

2. $\cos a = 0.8, \sin a = 0.6, \cos b = 0.6, \sin b = 0.8$

   $\therefore \cos(a+b) - \sin(a-b) = [(0.6)(0.8)-(0.6)(0.8)] - [(0.6)(0.6)-(0.8)(0.8)] =$

   $0 - [0.36-(0.64)] = \underline{\mathbf{0.28}}$ or $\underline{\mathbf{\frac{7}{25}}}$.

3. $\sin(x+17) = \cos(2x-23) \Leftrightarrow \sin(x+17) = \sin(90-(2x-23)) = \sin(113-2x)$.

   If $\sin A = \sin B$, then A and B are equal or supplementary. More specifically, $A = B + 360k$ or $A + B = 180 + 360k$, for any integer k.

   Adding $360k$ generates all coterminal angles.

   $x+17 = 113-2x+360k \Rightarrow 3x = 96+360k \Rightarrow x = 32+120k \Rightarrow$ 32, 152, 272 (for k = 0,1 and 2).

   $(x+17)+(113-2x) = 180+360k \Rightarrow 130-x = 180+360k$

   $\Rightarrow x = -50-360k \Rightarrow$ 310 (for $k = -1$).

   Thus, $(A,B,C,D) = \underline{\mathbf{(32, 152, 272, 310)}}$.

Team Round

1. Using the double angle identity $2\sin A\cos A = \sin 2A$ multiple times, we have

   $2\sin x\cos x\cos 2x\cos 4x\cos 8x \geq \frac{1}{16} \rightarrow \sin 2x\cos 2x\cos 4x\cos 8x \geq \frac{1}{16}$,

   $2\sin 2x\cos 2x\cos 4x\cos 8x \geq \frac{1}{16}\cdot 2 \rightarrow \sin 4x\cos 4x\cos 8x \geq \frac{1}{8}$,

   $2\sin 4x\cos 4x\cos 8x \geq \frac{1}{8}\cdot 2 \rightarrow \sin 8x\cos 8x \geq \frac{1}{4}$. Finally, $2\sin 8x\cos 8x \geq \frac{1}{4}\cdot 2$ gives

   $\sin 16x \geq \frac{1}{2}$. Setting $16x = \frac{\pi}{6}+2\pi k$ or $\frac{5\pi}{6}+2\pi k$ gives $x = \frac{\pi}{96}+\frac{\pi}{8}k$ or $\frac{5\pi}{96}+\frac{\pi}{8}k$. The largest possible value of x will occur at the second of the two values for $k = 3$ since when $k \geq 4$ the solutions are outside the domain. When $k = 3$, $x = \underline{\mathbf{\frac{41\pi}{96}}}$.

Team Round - continued

2. Since the equation of the parabola is $y=\dfrac{x^2}{4p}$ the focal point $F=(0,p)$. The coordinates of $M=\left(a,\dfrac{a^2}{4p}\right)$ and since ΔFOM is isosceles with $OF=OM$, then $p^2=a^2+\dfrac{a^4}{16p^2}$. Thus,

$$a^4+16p^2a^2-16p^4=0 \rightarrow a^2=\frac{-16p^2\pm\sqrt{256p^4+64p^4}}{2} \rightarrow$$

$$a^2=\frac{8p^2\sqrt{5}-16p^2}{2}=p^2\left(4\sqrt{5}-8\right).\ \text{Thus, } \frac{a^2}{p^2}=\boxed{4\sqrt{5}-8}.$$

3. $xy-2y+x=2 \Rightarrow y(x-2)=2-x \Rightarrow y=\dfrac{2-x}{x-2}=\dfrac{-1\cancel{(x-2)}}{\cancel{(x-2)}}=-1$, provided $x\neq 2$.

But remember there was no such restriction on the original equation! Substituting in the first equation, $x^2=2 \Rightarrow x=\pm\sqrt{2}$. However, if $x=2$, we have $y^2=3 \Rightarrow y=\pm\sqrt{3}$.
Thus, we have 4 ordered pairs, namely <u>$(\mathbf{2},\pm\sqrt{\mathbf{3}})$</u>, <u>$(\pm\sqrt{\mathbf{2}},-\mathbf{1})$</u>.

Team Round - continued

4. Assume the first sequence is A: $a, ar, ar^2, \ldots$ and the second B: $b, bm, bm^2, \ldots$. Then:

$$\left.\begin{array}{l}(1)\ a = bm^2 \\ (2) ar = bm\end{array}\right\} \Rightarrow \frac{1}{r} = m \quad \text{and} \quad \sum_1^7 A = 8 \cdot \sum_1^7 B \Leftrightarrow a\left(\frac{r^6-1}{r-1}\right) = 8b\left(\frac{m^6-1}{m-1}\right)$$

$$(3) b = ar^2$$

Substituting for b and m, $a\left(\frac{r^6-1}{r-1}\right) = 8ar^2\left(\frac{r^{-6}-1}{r^{-1}-1}\right) = 8ar^2\left(\frac{1-r^6}{r^5-r^6}\right) = 8ar^2\left(\frac{r^6-1}{r^5(r-1)}\right)$

$\Rightarrow 1 = \frac{8}{r^3} \Rightarrow r = 2, m = \frac{1}{2}$, but we stipulated that $r < m$. Our assignment of first and second was arbitrary, so we simply reverse the roles of r and m. $(r,m) = \underline{\left(\mathbf{\frac{1}{2}, 2}\right)}$.

Alternate Solution:

Sequence 1: $1, r, r^2, r^3, r^4, r^5$; sequence 2: $r^2, r, r^0, r^{-1}, r^{-2}, r^{-3}$

$S_1 = \frac{r^6-1}{r-1}$; $S_2 = \frac{\frac{r^6-1}{r^4}}{1-\frac{1}{r}} = \frac{r^6-1}{r^4} \cdot \frac{r}{r-1}$. Now $\frac{r^6-1}{r-1}$ cancels leaving $1 = \frac{8}{r^3} \rightarrow r = 2$.

5. There are ${}_{10}C_3 = \frac{10!}{3! \cdot 7!} = 120$ possible sets of triples. Given the particular numbers in S, the only sums divisible by 10 are 40 and 50. Here are the sets that sum to 40: $\{10,11,19\}$, $\{10,12,18\}, \{10,13,17\}, \{10,14,16\}, \{11,12,17\}, \{11,13,16\}$, $\{11,14,15\}$ and $\{12,13,15\}$. The sets that sum to 50 are: $\{13,18,19\}, \{14,17,19\}, \{15,16,19\}$ and $\{15,17,18\}$. There are a total of 12 sets, making the probability that the sum would be divisible by 10 equal to $\frac{12}{120} = \boxed{\frac{1}{10}}$.

6. The general term of the binomial expansion is $C_k^{12}\left(x^3\right)^{(12-k)} x^{-2k} = C_k^{12} x^{(36-5k)}$. So we have $36 - 5k = 11$, and $5k = 25$, so $k = 5$. Then $P = C(12, 5) = C(12, 7)$, and $n = 36 - 5(7) = 1$.
$P = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = (11)(9)(8) = 792$. So $(P, n) = \underline{\mathbf{(792, 1)}}$.