# MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

## NEW ENGLAND PLAYOFFS – 2013 - SOLUTIONS

### Round 1 Arithmetic and Number Theory

1. There are 1.6 ounces of protein in the box so $\frac{\$3.30}{16} = \$2.0625$ Thus, an ounce of protein costs $\boxed{\$2.06}$.

2. Total number: EEEEENNN gives 56. Through C: 2 times EEEENN for $2 \cdot \frac{6!}{4!2!} = 30$. Through D: EEEENN by 2 gives 30. From A to C to D to B: 2 x 4 x 2 = 16. Thus, $\boxed{56 - (30 + 30 - 16) = 12}$.

3. If $n$ ends in 2 then $n + 3$ ends in 5 so doubling it will result in a 0 at the end. If $n$ ends in 7, then $n + 3$ ends in 0 so doubling it ends in 0. Thus, each set of 10 numbers starting with [1,10] contains two values of $n$ with the desired condition. From 1 to 2010 we have 201 sets of ten numbers making 402 values of $n$ that give a result ending in 0. To that answer we must add 1 for 2012, making a total of $\boxed{403}$.

### Round 2 Algebra 1

1. $\frac{2}{3}m - 3 = -1,\ 2 + \frac{3}{5}n = 8 \rightarrow 2m - 9 = -3,\ 10 + 3n = 40 \rightarrow \boxed{m = 3, n = 10}$

2. $\frac{6x^{-1}+1}{12x^{-1}+2} = \frac{1}{2} \rightarrow \frac{\frac{6}{x}+1}{\frac{12}{x}+2} = \frac{6+x}{12+2x} = \frac{1}{2} \rightarrow 12+2x = 12+2x$. This is true for all $x$ except those that give 0 in the denominator, namely 0 and –6. Answer: all Reals except 0 and –6.

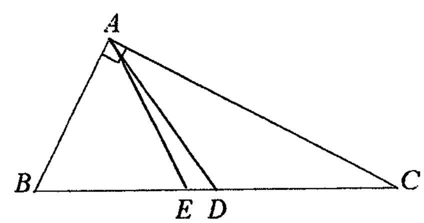
3. $1 \cdot b^4 + 0 \cdot b^3 + 1 \cdot b^2 + 0 \cdot b + 1 = 1 \cdot (2b)^2 + 0 \cdot (2b) + 1 \rightarrow b^4 + b^2 + 1 = 4b^2 + 1$. Simplifying gives $b^4 - 3b^2 = 0 \rightarrow b^2\left(b^2 - 3\right) = 0$. Thus, $\boxed{b = \sqrt{3}}$.

## Round 3 – Geometry

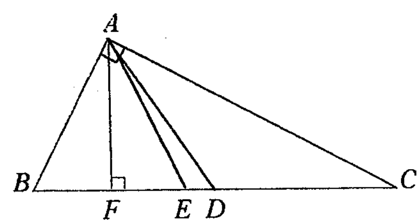
1. Since the angle measures must be integers, the value of $x$ which makes $3x + 10$ as small as possible is $-3$. If $x = -3$, then $3x + 10 = 1$, $\boxed{1°}$ is the smallest angle

2. Since a regular hexagon can be thought of as consisting of six equilateral triangles with a common vertex, the radius of the circle is the same as a side of the hexagon, i.e. 4. Therefore the area of the circle is $\boldsymbol{\pi \cdot 4^2 = 16\pi}$. Similarly the area of the hexagon is $\frac{3}{2} \cdot 4^2 \cdot \sqrt{3} = \mathbf{24\sqrt{3}}$. For the rectangle two of the sides coincide with sides of the hexagon, The other two sides are the bases of isosceles triangles of side 4 and vertex angle 120°. Drawing an altitude to the base creates two 30-60-90 triangles which leads to a base of $\mathbf{4\sqrt{3}}$. The area of the rectangle is $\mathbf{4 \cdot 4\sqrt{4} = 16\sqrt{3}}$. The required ratio is $\frac{16\pi - 24\sqrt{3}}{8\sqrt{3}} = \frac{2\pi - 3\sqrt{3}}{\sqrt{3}} = \boxed{\frac{2\pi\sqrt{3} - 9}{3}}$
Note: The answer is independent of the length of the side of the hexagon..

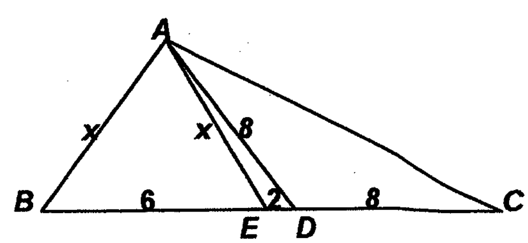
3. Since $ABC$ is a right triangle and $D$ is the midpoint of the hypotenuse, $BD = AD$, so $\angle B \cong \angle BAD$. It is given that $AB = AE$ so $\angle B \cong \angle AEB$. Thus $\Delta ABD \sim \Delta BEA$ giving $\frac{AB}{BE} = \frac{AD}{AB}$ giving $AB^2 = BE \cdot AD$. Since $DC = 8$ and $D$ is the midpoint of $\overline{BC}$, then $BE = 6$ giving $AB^2 = 6 \cdot 8$. Thus $\boxed{AB = 4\sqrt{3}}$.




Alternate solution: Drop the altitude from $A$. Note that $BF = FE$ since $\Delta ABE$ is isosceles. By the geometric mean theorem, $AB^2 = BF \cdot BC = \frac{1}{2}BE \cdot 2DC = BE \cdot DC$. Since $DC = 8$ and $D$ is the midpoint of $\overline{BC}$, then $BE = 6$ giving $AB^2 = 6 \cdot 8$. Thus $AB = 4\sqrt{3}$.




Alternate Solution 2: Using Stewart's theorem on $\Delta BAD$, we have $x^2 \cdot 2 + 8^2 \cdot 6 = x^2 \cdot 8 + 6 \cdot 2 \cdot 8$
$\Rightarrow 8^2 \cdot 6 - 6 \cdot 2 \cdot 8 = 6x^2 \Rightarrow 64 - 16 = x^2 \Rightarrow x = \boxed{4\sqrt{3}}$



**Round 4 – Algebra 2**

1. Basically, $x = \frac{1-b}{a}$ and checking all possible combinations of $a$ and $b$ gives 13 distinct values for $x$. Some values of $a$ and $b$ give the same value for $x$, namely (3, 4), (4, 5), and (2, 3) all give –1, and (2, 2) and (4, 3) give –1/2. So out of the 16 possible combinations of $a$ and $b$, we reject 3, giving $\boxed{13}$.

2. $\log 2013^a = \log 100 = 2 \rightarrow a = \frac{2}{\log 2013} \rightarrow \frac{1}{a} = \frac{\log 2013}{2}$. Similarly, $\log .2013^b = 100$ gives $\frac{1}{b} = \frac{\log .2013}{2}$. So. $\frac{1}{a} - \frac{1}{b} = \frac{\log 2013 - \log .2013}{2} = \frac{\log \frac{2013}{.2013}}{2} = \frac{\log 10^4}{2} = \boxed{2}$.

Alternate Solution: The solution is independent to the bases.

$10^b = 100 \rightarrow b = 2$; $[10{,}000 \cdot 10]^a = 100 \rightarrow a = \frac{2}{5}$. $\frac{1}{a} - \frac{1}{b} = \frac{5}{2} - \frac{1}{2} = 2$

3. Solve $x + ry = r^2$ and $x + ty = t^2$ by subtracting to obtain $(r-t)y = r^2 - t^2 \rightarrow$ $y = r + t$ and $x + r(r+t) = r^2 \rightarrow x = -rt$. With $r = \frac{7-\sqrt{51}}{2}$ and $t = \frac{7+\sqrt{51}}{2}$, we obtain the ordered pair $\boxed{\left(\frac{1}{2}, 7\right)}$.

**Round 5 – Analytic Geometry**

1. Since $O$ and $C$ are both equidistant from the endpoints of segment $\overline{AB}$, $\overline{OC}$ is the perpendicular bisector of $\overline{AB}$ and $m\angle COB = 30$ so the slope of $\overline{OC}$ = tan 30 = $\boxed{\frac{\sqrt{3}}{3}}$.

2. The center of the circle is at $(3, -4)$with radius 2. The lower end of the vertical diameter and hence the vertex of the parabola is $(3, -6)$. One end of the major axis is $(0, -4)$, the y-intercept of the graph of the linear equation. By symmetry, the other

end is $(6,-4)$. Substituting these points in to $y = ax^2 + bx + c$ gives three equations $-6 = 9a + 3b + c$, $-4 = 0a + 0b + c$, and $-4 = 36a + 6b + c$. The second equation gives $c = -4$. Substituting this into the other two equations and solving them as a linear system of two equations in two variables gives $a = \frac{2}{9}$ and $b = -\frac{4}{3}$. The ordered triple is $\boxed{\left(\frac{2}{9}, -\frac{4}{3}, -4\right)}$

Alternate Solution: The vertex of the parabola is (3, –6) and it contains (0, –4), so the equation is $y + 6 = a(x-3)^2$. Substituting $(0,-4)$ for x and y gives $a = \frac{2}{9}$.

3. By the symmetry of the situation the center of the circle would be $Q(a,a)$, the radius would have length $a$, and the tangent point of intersection with $xy = 2$ would be $T\left(\sqrt{2},\sqrt{2}\right)$. Then $\left(a-\sqrt{2}\right)^2 + \left(a-\sqrt{2}\right)^2 = a^2 \rightarrow a^2 - 4a\sqrt{2} + 4 = 0$. Solving gives $a = 2\sqrt{2} \pm 2$. The larger value lies above the graph so we choose $a = 2\sqrt{2} - 2$. The sum of the coordinates of the center is $\boxed{4\sqrt{2} - 4}$.

Alternate Solution: With $Q = (a,a)$ and $T = (\sqrt{2},\sqrt{2})$, we have $TO = a + a\sqrt{2} = 2 \rightarrow$ $a = \frac{2}{\sqrt{2}+1} = 2\sqrt{2} - 2 \rightarrow 2a = 4\sqrt{2} - 4$

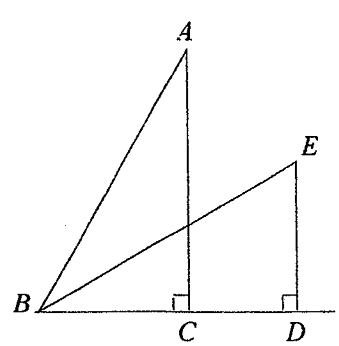
## Round 6 – Trig and Complex Numbers

1. $\left[4\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right]^2 \cdot 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 16\left(\frac{1}{2} + i - \frac{1}{2}\right)\left(-\sqrt{3} + i\right) = 16i(-\sqrt{3} + i) =$ $\boxed{-16 - 16i\sqrt{3}}$

Alternate Solution:

$[(4cis45)^2][2cis150] = [16cis90][2\ cis\ 150] = 32cis240 = -16 - 16i\sqrt{3}$.

2. Let the length of the pole be $x$. Then $ED = \frac{x}{2}$ and $BD = \frac{x}{2}\sqrt{3}$. Also, $BC = \frac{x}{2}$. Since $CD = 20$, then $BD - BC = 20 \rightarrow \frac{x}{2}\sqrt{3} - \frac{x}{2} = 20$. Then $x = \frac{40}{\sqrt{3}-1} = \boxed{20\sqrt{3} + 20}$.



3. Since $\Delta APD \cong \Delta BPC$, the area of $\Delta BPC$ is 9. Since triangles $PAB$ and $PCD$ are equilateral, then $m\angle BPC = 120°$, giving $\frac{1}{2} \cdot BP \cdot PC \cdot \sin 120 = 9$. Thus, $BP \cdot PC = 18 \cdot \frac{2}{\sqrt{3}} = \boxed{12\sqrt{3}}$.

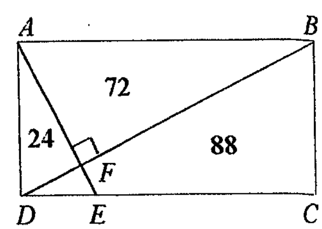
## Team Round

1. From $4 - 2^a = 2^c - 4$ we obtain $8 = 2^a + 2^c$. The value of $c$ is as large as possible when $2^a$ is as small as possible, so let $a = 1$, making $2^c = 6$ giving $\boxed{c = \log_2 6}$.

2. Let $x = 1000$, giving $(x+3)^3 - 3(x+1)^3 + 3(x-1)^3 - (x-3)^3$. The first and last terms sum to $(x^3 + 9x^2 + 27x + 27) - (x^3 - 9x^2 + 27x - 27) = 18x^2 + 54$. The second and third terms sum to $3(x^3 - 3x^2 + 3x - 1) - 3(x^3 + 3x^2 + 3x + 1) = -18x^2 - 6$. Adding this to $18x^2 + 54$ gives $\boxed{48}$. The sum is invariant and does not depend on $x$.

3. $M$ can't be greater than or equal to 5. If $M = 1$ we have $3 = \frac{51}{17}$ or $7 = \frac{91}{13}$, if $M = 2$, $2 = \frac{42}{21}$ and $2 = \frac{52}{26}$ but both fail since $M = I$. However, $3 = \frac{72}{24}$ and $4 = \frac{92}{23}$ both work. If $M = 3$ we have $3 = \frac{93}{31}$ which fails since $M = I$. If $M = 4$, $2 = \frac{84}{42}$ fails since $E = I$, but $2 = \frac{94}{47}$ works. Thus, the solutions are $2 = \frac{94}{47}$, $3 = \frac{51}{17}$, $7 = \frac{91}{13}$, $3 = \frac{72}{24}$, $4 = \frac{92}{23}$. So $I$ takes on the values of $\boxed{2, 3, 4, \text{and } 7}$.

4. The area of $\Delta EDF = (24 + 72) - 88 = 8$. Since $\Delta ADF$ and $\Delta EDF$ have the same height, the ratio of their areas equals the ratio of their bases so $\frac{AF}{EF} = 3$. Let $AF = 3x$ and $EF = x$. Similarly, let $DF = y$ and $BF = 3y$. Then




$AE \cdot DB = (4x)(4y)$. Since the area of $\Delta DFE = \frac{1}{2} \cdot x \cdot y = \frac{xy}{2} = 8$, then $xy = 16$. This makes $AE \cdot DB = (4 \cdot 4)xy = 16 \cdot 16 = \boxed{256}$. One can solve for the lengths and obtain $FE = \frac{4}{\sqrt[4]{3}}$, $FA = \frac{12}{\sqrt[4]{3}}$, $DF = 4\sqrt[4]{3}$, and $BF = 12\sqrt[4]{3}$,

5. Expanding $\left(z^2+\frac{1}{z^2}\right)^2+\left(z+\frac{1}{z}\right)^2=4$ we obtain $z^4+2+\frac{1}{z^4}+z^2+2+\frac{1}{z^2}=4$.

Subtracting 4 and multiplying by $z^4$ gives $z^8+z^6+z^2+1=0$. This factors as $z^6\left(z^2+1\right)+\left(z^2+1\right)=0\rightarrow\left(z^6+1\right)\left(z^2+1\right)=0$. The six solutions to $z^6=-1$ form a hexagon of radius 1 centered at the origin. The two solutions to $z^2=-1$ are also solutions to the first equation since $\left(z^2\right)^3=z^6=(-1)^3=-1$ so they don't add any vertices. Thus the area is the area of the hexagon which is $6\cdot\frac{1^2\sqrt{3}}{4}=\boxed{\frac{3\sqrt{3}}{2}}$.

6. Let $OA=3a$ and $OB=3b$, making $AB=3\sqrt{a^2+b^2}$. Since $PA=\sqrt{a^2+b^2}$, then $PT=b$, $AT=a$, $OT=2a$, making the slope of $\overline{OP}=\frac{b}{2a}$. Since $QA=2\sqrt{a^2+b^2}$, $QR=2b$, $AR=2a$, $OR=a$, making the slope of $\overline{OQ}=\frac{2b}{a}$. The product of the slopes is $\frac{b^2}{a^2}$. The slope of $\overline{AB}=-\frac{3b}{3a}=-\frac{b}{a}=k$. Thus, the product of the slopes equals $\boxed{k^2}$.