MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2011 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. $3(1.\overline{2})(.\overline{20}) = 3 \cdot \frac{11}{9} \cdot \frac{20}{99} = \frac{20}{27} = .\overline{740}$. Thus, $(A, B, C) = \boxed{(7, 4, 0)}$.

2. If a number has exactly three factors, then it must be of the form p^2, where p is a prime. The answer is $97^2 = \boxed{9409}$.

3. Since $N = \frac{10A + B}{99}$ and $M = \frac{10B + A}{99}$ we have $\frac{4}{7}(10A + B) = (10B + A) \rightarrow A = 2B$.
Thus, $N = .\overline{21}, .\overline{42}, .\overline{63}$, and $.\overline{84}$. In reduced fractional form $\boxed{N = \frac{7}{33}, \frac{14}{33}, \frac{7}{11}, \frac{28}{33}}$.

Round 2 Algebra 1

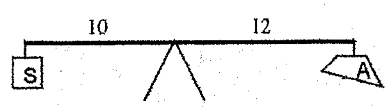
1. Let x = the number of points he earned and let y = the total number of possible points. Then $\frac{x}{y} = .75$ and $\frac{x+6}{y} = .76$ giving $.75y + 6 = .76y \rightarrow 6 = .01y$. Thus, $y = \boxed{600}$.
Alternate Solution: 6 points is equivalent to a 1% improvement. $\therefore$600 points = 100%,

2. We need at least 4 elements and that would give $\frac{1+1+2+x}{4} = 3$, making $x = 8$, but the value of x that gives the right mean gives a median of 1.5, not 2. Consider a set of five elements: 1, 1, 2, x, y. Then $\frac{1+1+2+x+y}{5} = 3$, giving $x + y = 11$. A set with more elements will have a larger sum, so the answer must have five elements. The set with the least largest element would be $\boxed{\{1, 1, 2, 5, 6\}}$.

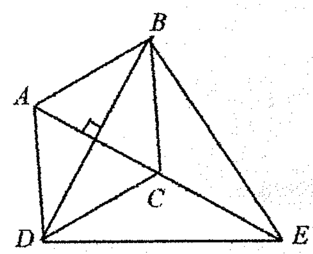
3. Simplifying gives $\sqrt{\frac{(x-4)^2}{(x-3)^2}} = \frac{1}{2} \rightarrow \left|\frac{x-4}{x-3}\right| = \frac{1}{2}$. If $x \geq 4$ we have $\frac{x-4}{x-3} = \frac{1}{2} \rightarrow x = 5$. If $3 < x < 4$, we have $\frac{4-x}{x-3} = \frac{1}{2} \rightarrow x = \frac{11}{3}$. If $x < 3$, we have $\frac{4-x}{3-x} = \frac{1}{2} \rightarrow x = 5$ but this lies outside the range of values in this case. Hence, $\boxed{x = 5, \frac{11}{3}}$.
Alternate Solution: Express the radicand as a single fraction, square both sides, and cross multiply. $4\left[(x-3)^2 - 2x + 7\right] = (x-3)^2 \rightarrow 3x^2 - 26x + 55 = (3x - 11)(x - 3) = 0$.

Round 3 – Geometry

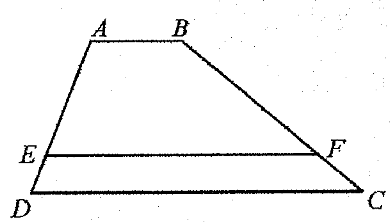
1. Let W_A and W_S be the weights of A and S respectively. Then $12 \cdot W_A = 10 \cdot W_S$, making $\frac{W_A}{W_S} = \frac{5}{6}$. Since the figures are made of the same material, the ratio of weights is the same as the ratio of their areas so the area of A is $\frac{5}{6} \cdot 16 = \boxed{\frac{40}{3}}$.



2. Since $\overline{BD} \perp \overline{AE}$, ΔBAC and ΔBAE have the same altitude so their areas have the same ratio as their bases, namely $1:3$. Thus, the ratio of the area of $ABCD$ to the area of $ABED$ is $\boxed{\frac{1}{3}}$.



3. Let $AE = 4x$, $ED = x$, $BF = 4y$, and $FC = y$. Then $4x + 4y + 10 + EF = x + y + 30 + EF$. So $x + y = \frac{20}{3} \to 5x + 5y = AD + BC = \frac{100}{3}$. The perimeter of $ABCD = 40 + \frac{100}{3} = \boxed{\frac{220}{3}}$.



Round 4 – Algebra 2

1. $a = 4\log 4 \to \frac{a}{4} = \log 4$; $b = 5\log 5 \to \frac{b}{5} = \log 5$. Also, $\log 20^{20} = 20(\log 4 + \log 5)$, giving $20\left(\frac{a}{4} + \frac{b}{5}\right) = \boxed{5a + 4b}$.

2. There are ${}_{20}C_3 = \frac{20!}{3! \cdot 17!} = 1140$ possible unordered triples of numbers. Of those the following form an increasing geometric sequence:

a_1	1	1	1	2	2	3	4	4	5	8	9
r	2	3	4	2	3	2	3/2	2	2	3/2	4/3
Seq	1,2,4	1,3,9	1,4,16	2,4,8	2,6,18	3,6,12	4, 6, 9	4,8,16	5,10,20	8,12,18	9,12,16

There are 11 sequences so the probability is $\boxed{\frac{11}{1140}}$.

3. Let $x = 1000$, giving $(x+3)^3 - 4(x+1)^3 + 4(x-1)^3 - (x-3)^3$. The first and last terms sum to $(x^3 + 9x^2 + 27x + 27) - (x^3 - 9x^2 + 27x - 27) = 18x^2 + 54$. The second and third terms sum to $4(x^3 - 3x^2 + 3x - 1) - 4(x^3 + 3x^2 + 3x + 1) = -24x^2 - 8$. Adding this to $18x^2 + 54$ gives $-6x^2 + 46$. Since $x = 1000$, the sum is $\boxed{-5{,}999{,}954}$.

   Note: in this case the sum depends upon the value of x. But, if we replace the 4's with 3's and proceed as above, we obtain $(x+3)^3 - 3(x+1)^3 + 3(x-1)^3 - (x-3)^3 =$ $\left(18x^2 + 54\right) + \left(-18x^2 - 6\right) = 48$. In this case, the sum does not depend on the value of x, that is, it is invariant.

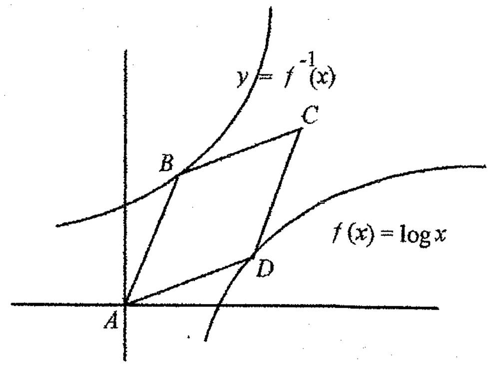
Round 5

Round 5 – Analytic Geometry

1. The intersection points of the parabola and horizontal line are $(2,-10)$ and $(6,-10)$. The minor axis is 4 units long. $y = -3x^2 + 24x - 46 = -3(x-4)^2 + 2$ The vertex of the parabola is at $(4,2)$. The other end of the major axis is at $(4,-22)$ making it 24 units long. $24 + 4 = \boxed{28}$

2. The trisection points are $D = (3,1)$ and $C = (6,2)$. Let $P = (0,y)$. The absolute value of the slope of $\overline{PC}$ is less than the absolute value of the slope of $\overline{PD}$ so $\frac{m\overline{PC}}{m\overline{PD}} = \frac{1}{6} \rightarrow \frac{y-2}{-6} \Big/ \frac{y-1}{-3} = \frac{1}{6}$. Thus, $\frac{y-2}{y-1} = \frac{1}{3}$ gives $y = \frac{5}{2}$. Answer: $\boxed{\left(0, \frac{5}{2}\right)}$.

3. Let the coordinates of D be $(x, \log x)$, making the coordinates of $B = (\log x, x)$. Set $AB = BC$ obtaining $\sqrt{(\log x)^2 + x^2} = \sqrt{(\log x - 4)^2 + (x-4)^2}$. Squaring and canceling gives $\log x + x = 4$. Ans: $\boxed{4}$ Note: Nothing in the solution requires logs. So the result holds as long as the problem involves a function and its inverse.



Alternate solution: The midpoint of $\overline{AC}$ is (2, 2), its slope is 1 and its equation is $x - y = 0$. Therefore the slope of $\overline{BD}$ is -1 and its equation is $x + y = 4$.

Alternate Solution #2

If the coordinates of D are (a, b), then the coordinates of B are (b, a). Since $ABCD$ is a rhombus, $AB^2 = BC^2 \rightarrow a^2 + b^2 = (4-a)^2 + (4-b)^2 \rightarrow 0 = 32 - 8a - 8b \rightarrow a + b = \boxed{4}$.

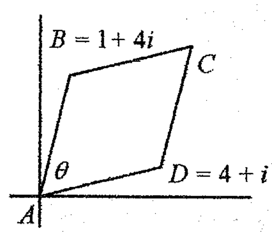
Round 6 – Trig and Complex Numbers

1. $\text{Tan}^{-1} -\frac{6}{5} = A \rightarrow \left(\frac{5}{\sqrt{61}}, -\frac{6}{\sqrt{61}}\right)$ is on the unit circle. $\therefore \sin 2A = 2\left(-\frac{6}{\sqrt{61}}\right)\left(\frac{5}{\sqrt{61}}\right) = \boxed{-\frac{60}{61}}$

2. $r = \dfrac{4}{2\sin\theta - 3\cos\theta} \rightarrow r = \dfrac{4}{2 \cdot \frac{y}{r} - 3 \cdot \frac{x}{r}} \rightarrow 2y - 3x = 4$.

   If $x = 5$ then $\boxed{y = \frac{19}{2}}$.

3. Using vectors, $B = \langle 1,4 \rangle, D = \langle 4,1 \rangle$, we obtain $\cos\angle\theta$ using the dot product: $\cos\angle\theta = \dfrac{\langle 1,4 \rangle \cdot \langle 4,1 \rangle}{\sqrt{17} \cdot \sqrt{17}} = \dfrac{8}{17}$. Then $\sin\angle\theta = \dfrac{15}{17}$, making the area of $\Delta BAD =$ $\frac{1}{2} \cdot \sqrt{17} \cdot \sqrt{17} \cdot \sin\theta = \frac{17}{2} \cdot \frac{15}{17} = \frac{15}{2}$. Thus, the area of the parallelogram is $\boxed{15}$.



Alternate Solution 1: By determinants $\begin{vmatrix} 4 & 1 \\ 5 & 5 \end{vmatrix} = 15$

Alternate solution 2: The equation of $\overline{AD}$ is $x - 4y = 0$ and the equation of $\overline{BC}$ is $x - 4y = -15$. The area is the difference of the C's: 0 – (–15) = 15.

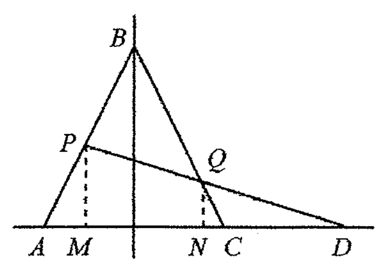
Alternate Solution 3: Alternate Solution 3:
C(5, 5) Listing the coordinates in clockwise (or counterclockwise) order starting at any vertex, the area is equal to

$$\frac{1}{2}\begin{vmatrix} 0 & 0 \\ 1 & 4 \\ 5 & 5 \\ 4 & 1 \\ 0 & 0 \end{vmatrix} = \frac{1}{2}\left|(0\cdot4+1\cdot5+5\cdot1+4\cdot0)-(0\cdot1+4\cdot5+5\cdot4+1\cdot0)\right| = \frac{1}{2}|10-40| = \boxed{15}$$

Team Round

1. Consider the remainder if we divide the sum of the digits by 9. That remainder is invariant. Dividing each of the numbers from 1 to 9, 10 to 18, etc., by 9 yields the nine remainders from 0 to 8, so for every set of consecutive nine numbers, the resulting one-digit number will be the numbers from 0 to 8. So from 1 to 999,999 there will be equal numbers of one-digit numbers from 0 to 8, namely 111,111 of each. Since the remainder when 10^6 is divided by 9 is 1, there is one more 1 among the one-digit numbers. Answer: $\boxed{111{,}112}$.

2. The equation of $\overline{AB}$ is $y = 3x + 6$, the equation of $\overline{BC}$ is $y = -3x + 6$, and the equation of the line through D with slope m is $y = m(x - 4)$. Setting the equations equal we find that $P = \left(\frac{4m+6}{m-3}, \frac{18m}{m-3}\right)$ and also $Q = \left(\frac{4m+6}{m+3}, \frac{-6m}{m+3}\right)$. The area of $\Delta ABC = \frac{1}{2}\cdot4\cdot6 = 12$. To find m, subtract the area of ΔQCD from the area of ΔPAD and set the result equal to 6. Thus, we have $\frac{1}{2}\cdot6\cdot\frac{18m}{m-3} - \frac{1}{2}\cdot2\cdot\frac{-6m}{m+3} = 6 \to 3m^2 + 8m + 3 = 0 \to m = \frac{-4\pm\sqrt{7}}{3}$. Since the slope of $\overline{BD}$ is $-\frac{3}{2}$, we choose the value of m giving a line not as steep as $\overline{BD}$: $\boxed{m = \frac{-4+\sqrt{7}}{3}}$.

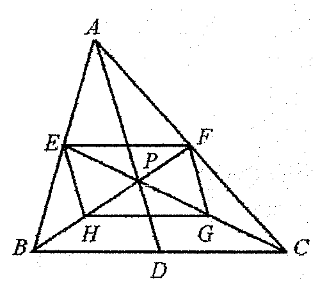


3. From $\frac{a}{5}+\frac{b}{5}+\frac{c}{5}=1$ we have $a+b+c=5$ where $a, b, c \in \{0, 1, 2, 3, 4, 5\}$. Thus, it seems that our answer will be partitions involving 3 numbers and the permutations of those triples.

Triples	number of permutations	Total
0-0-5	3	3
0-1-4	6	6
0-2-3	6	6
1-1-3	3	3
1-2-2	3	3

Makes $\boxed{21}$ different colors.

4. Since $\overline{EF}$ is a midline of ABC and $\overline{GH}$ is a midline of PBC, then both are parallel to $\overline{BC}$ and half as long. Thus, $EFGH$ is a parallelogram and $EF + GH = 42$. Since $\overline{EH}$ is a midline of BPA and $\overline{FG}$ is a midline of CPA, then $EH + FG = AP = \frac{2}{3}AD$. To calculate AD use the parallelogram law: $2\left(AB^2 + AC^2\right) = BC^2 + (2AD)^2 \rightarrow 2(21^2 + 27^2) = 42^2 + 4 \cdot AD^2$ giving $AD = 12$, so $AP = 8$. The perimeter of $EFGH = \boxed{50}$.



Alternate Solution: The length of median $\overline{AD}$ can be found from the median formula $\frac{1}{2}\sqrt{2 \cdot AB^2 + 2 \cdot AC^2 - BC^2}$. In this case $\frac{1}{2}\sqrt{2 \cdot 441 + 2 \cdot 729 - 1764} = 12$

Alternate Solution 2: Stewart's Theorem $\left(AB^2 \cdot BD + AC^2 \cdot BD = AD^2 \cdot BC + BC \cdot BD \cdot DC\right)$

$21^2 \cdot 21 + 27^2 \cdot 21 = AD^2 \cdot 42 + 42 \cdot 21^2 \rightarrow 21^2 + 27^2 = 2\left(AD^2 + 21^2\right)$

$\rightarrow 2AD^2 = 27^2 - 21 = (27+21)(27-21) = 48 \cdot 6$

$\rightarrow AD = \sqrt{48 \cdot 3} = \sqrt{16 \cdot 9} = 12$

... and the results follows.

5. Note that the difference between N and its percent P goes up as N increases. For example, if $N = 12, P = 10$ and if $N = 42, P = 35$. Setting $\frac{N}{120} = \frac{N - 4.5}{100}$ we obtain $N = 27$. Setting $\frac{N}{120} = \frac{N - 5.5}{100}$, we obtain $N = 33$. It would seem that for N

= 27, 28, 29, 30, 31, 32, and 33, then $P = N - 5$, but we must exclude 27 since 22.5 would round up to 23, a difference of 4. We don't exclude 33 since 27.5 rounds up to 28, a difference of 5. Thus, there are $\boxed{6}$ values of N for which the raw score exceeds the percentage by 5.

Alternate solution: Let $R(x)$ denote the rounding function. The percentage is obtained as $R\left(\frac{100N}{120}\right) = R\left(\frac{5N}{6}\right)$. We seek all N for which

$N - R\left(\frac{5N}{6}\right) = 5 \rightarrow N = R\left(\frac{5N}{6}\right) + 5$. By the nature of $R(x)$,

$$\frac{5N}{6} + 4.5 < R\left(\frac{5N}{6}\right) + 5 \le \frac{5N}{6} + 5.5 \rightarrow$$

$$\frac{5N}{6} + 4.5 < N \le \frac{5N}{6} + 5.5 \rightarrow 27 < N \le 33 .$$

6. Convert to polar:

$$\left(x^2 + y^2\right)^3 = 9x^2y^2 \rightarrow \left(r^2\right)^3 = 9(r\cos\theta)^2(r\sin\theta)^2 \rightarrow r^2 = 9\cos^2\theta\sin^2\theta.$$

Then $r = \pm 3\cos\theta\sin\theta \rightarrow r = \pm\frac{3}{2}\cdot 2\cos\theta\sin\theta = \pm\frac{3}{2}\sin 2\theta$. The maximum distance of a point on this curve from the pole or origin is $\boxed{\frac{3}{2}}$.