MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2008 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. The numbers are 98, 99, 100. The divisors are 49, 9, 100. The quotients are 2, 11, 1

2. $101^3 - 99^3 = (101-99)(101^2 + 101 \cdot 99 + 99^2) = 2\left((101+99)^2 - 101 \cdot 99\right) =$ $2(40{,}000 - 9999) = 2(30{,}001) = 60{,}002$. Divide by 2 to get 30,001 cubic centimeters.

3. $p + 2p + 3p + \cdots + 24p = \frac{(p+24p)}{2} \cdot 24 = 12 \cdot 25p = 2^2 \cdot 3 \cdot 5^2 p$. If $p = 3$, there are 3(3)3 = 27 factors. If $p = 2$ or 5 there are 4(2)3 = 24 factors. If p equals a prime other than 2, 3, or 5, there are 3(2)(3)2 = 36 factors. So $p = 7$.

Round 2 Algebra 1

1. Let the numbers be denoted by $n-2$, $n-1$, n, and $n+1$. Then $(n-2)(n+1) = n^2 - 55$

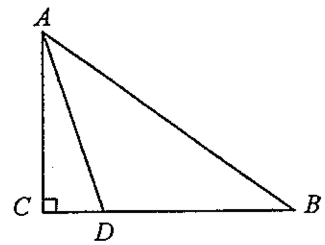
2. Let x be the number of strips added. Then for each shaded square we add three new squares. Thus, $\frac{7+x}{15+3x} < \frac{2}{5} \rightarrow 35 + 5x < 30 + 6x \rightarrow x > 5$. Therefore, $x = 6$.

3. Adding the equations we obtain $2008x + 2008y = 2008 \rightarrow x + y = 1$. Substituting $y = 1 - x$ into the first equation gives $1003x + 1001 - 1001x = 817 \rightarrow 2x = -184$; thus, $x = -92$.

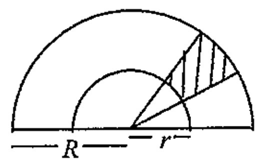
Round 3 – Geometry

1. Let $ED = x$, then $CE = 3x + 1$. $x(3x+1) = 80$

2. Let $CD = 3x$, then $AD = 5x$ and $AC = 4x$, giving $\rightarrow$
$AB^2 = (4x)^2 + (8x)^2$



3. Thinking of two complete circles, there are 13 additional regions congruent to the shaded one so $\dfrac{\pi R^2 - \pi r^2}{14} = 12$.



From $R = 2r$ we have $\pi\left(R^2 - \dfrac{R^2}{4}\right) = 14 \cdot 12 \rightarrow$

$\pi R^2 = 14 \cdot 12 \cdot \dfrac{4}{3} = 224$. The area of the large semicircle equals $\dfrac{\pi R^2}{2} = 112$.

Round 4 – Algebra 2

1. The function is quadratic in $\log_7 x$, so the minimum occurs at $\log_7 x = -\dfrac{-4}{2} = 2$. Thus, the low point is (49, –6).

2. Expanding gives $x^5 + 200 - 8x^2 - 25x^3$. Rearranging and factoring gives $(x^3 - 8)(x^2 - 25)$.

3. Let the height of all the triangles be a and the base of the leftmost triangle be $2b$. Then $ab = 9$. If the slope of the left hand side of the second triangle is twice the slope of the left hand side of the first triangle, then the base of the second triangle must be b giving an area of $\dfrac{1}{2}ba = \dfrac{9}{2}$. Similarly, the base of the third triangle must be $\dfrac{b}{2}$ giving an area of $\dfrac{1}{2} \cdot \dfrac{b}{2} \cdot a = \dfrac{ab}{4} = \dfrac{9}{4}$. We see

that the areas form an infinite geometric series whose first term is 9 and whose common ratio is $\frac{1}{2}$. The sum of the areas $\frac{9}{1-1/2} = 18$.

Round 5 – Analytic Geometry

1. $\frac{-a^2}{2-a} = -1$

2. $\sqrt{(f(1)-1)^2 + (1-f(1))^2} = (f(1)-1)\sqrt{2} = \sqrt{6}$, so $f(1) - 1 = \sqrt{3}$ giving $f(1) = \sqrt{3} + 1$. Since $f(x) = mx$, then $f(1) = m$, so the slope is $\sqrt{3} + 1$.

3. $y = a(x-2)^2 - 9 = 0 \rightarrow x = 2 \pm \frac{3}{\sqrt{a}}$. Setting $2 - \frac{3}{\sqrt{a}} \geq 0 \rightarrow \sqrt{a} \geq \frac{3}{2}$, so $a \geq \frac{9}{4}$.

Round 6 – Trig and Complex Numbers

1. $\sin A = \frac{3}{5}$, $\cos A = -\frac{4}{5}$, $\sin B = \frac{5}{13}$, $\cos B = \frac{12}{13}$

2. $(p+q)^2 = p^2 + 2pq + q^2$ and $p^2q^2 = 9 + 7 = 16$ so $pq = \pm 4$. If $pq = -4$ then $(p+q)^2 = 6 - 8$ and that can't be. So $pq = 4$ then $(p+q)^2 = 6 + 8 = 14$. Thus, $p + q = \sqrt{14}$.

3. $z_1 = 2^{3/2} cis \frac{\pi}{4}$ and $w_0 = 2^{1/10} cis \frac{7\pi}{20}$.

Team Round

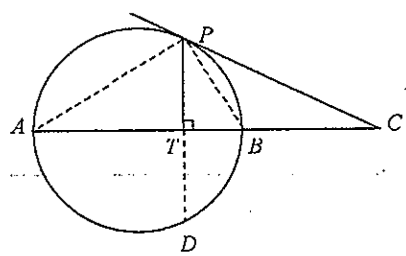
1. Let $B = \left(x, \frac{x^2}{3}\right)$. The center of the circle is $C = \left(\frac{x+1}{2}, \frac{x^2+1}{6}\right)$. Since the circle is tangent to the x-axis, set the radius equal to the y-value of C and solve for x:

$$\frac{1}{2}\sqrt{(x-1)^2+\left(\frac{x^2}{3}-\frac{1}{3}\right)^2}=\frac{x^2+1}{6} \to 9(x-1)^2+9\left(\frac{x^2-1}{3}\right)^2=\left(x^2+1\right)^2 \to$$

$5x^2-18x+9=0 \to (5x-3)(x-3)=0 \to x=\frac{3}{5}$ or 3. Thus, B could be either $\left(\frac{3}{5},\frac{3}{25}\right)$ or $(3,3)$.

2. Let t be the time in terms of revolutions traveled by the minute hand, making $\frac{t}{12}$ the time in revolutions traveled by the hour hand. At 3:00 the two hands form a 90° angle. Thereafter, the angle decreases to 0 and then increases once the minute hand passes the hour hand. They again form a 90° angle at some time past 3:30. From that point on until 4:00 they form an obtuse angle. Solving $t-\frac{t}{12}=\frac{1}{2}$, we obtain the time it takes for the minute hand to get 90° past the hour hand. Here, $t=\frac{6}{11}$. Thus, for the remaining $\frac{5}{11}$ hour the hands form an obtuse angle. The probability is $\frac{5}{11}$.

3. Since ΔAPB is a right triangle then $PT^2=AT\cdot TB=96$. Or, extend $\overline{PT}$ to D and by the Power of the Point Theorem, $AT\cdot TB=PT\cdot TD=PT^2$. Since



$CT^2=PC^2-PT^2$ and $PC^2=CA\cdot CB=(CT+TA)(CT-TB)=(CT+12)(CT-8)$, then $CT^2=(CT+12)(CT-8)-96=CT^2+4(CT)-96-96$. Thus we have $4\cdot CT=192 \to CT=48$.

4. If the leftmost digit is a 1, it may be followed by any one of 8 digits. The next three digits may be any three of the 10 digits, but in just one order, given by ${}_{10}C_3$. If the leftmost digit is a 2, it may be followed by any one of 7 digits, but the number of possibilities for the next three digits is again given by ${}_{10}C_3$. The counting continues in similar fashion yielding the following total:

$8\left({}_{10}C_3\right) + 7\left({}_{10}C_3\right) + 6\left({}_{10}C_3\right) + 5\left({}_{10}C_3\right) + \ldots + 1\left({}_{10}C_3\right) = (1 + 2 + \ldots + 8)\left({}_{10}C_3\right) = \frac{(1+8)8}{2}\frac{10!}{3!\,7!} = 36 \cdot 5 \cdot 3 \cdot 8 = 4320.$

5. From $f_1(x) = \frac{1}{x-1}$, $f_2(x) = \left(1 + \frac{1}{x-1}\right)^{-1} = \frac{x-1}{x}$, $f_3(x) = \left(1 + \frac{x-1}{x}\right)^{-1} = \frac{x}{2x-1}$, $f_4(x) = \left(1 + \frac{x}{2x-1}\right)^{-1} = \frac{2x-1}{3x-1}$, and $f_5(x) = \left(1 + \frac{2x-1}{3x-1}\right)^{-1} = \frac{3x-1}{5x-2}$, we see that $g_5(x) = \frac{1}{5x-2}$ since all other terms cancel. Thus, $g_5(2) = \frac{1}{8}$.

6. Since $a_n = 3 + 9(n-1)$ and $b_m = 4 + 11(m-1)$ then $a_n - b_m = (3 + 9n - 9) - (4 + 11m - 11) = 9n - 11m + 1$. From $9n - 11m + 1 = 1$ we obtain $9n = 11m \rightarrow n = \frac{11m}{9}$, giving $(n,m) = (11,9),(22,18),\ldots,(99,81)$ for a total of 9 ordered pairs. Now consider $b_m - a_n = 11m - 9n - 1 = 1 \rightarrow 11m - 9n = 2$. Solving for n, we obtain $n = m + \frac{2m-2}{9}$. Letting $m = 9t + 1$, we obtain $n = 11t + 1$. For $t = 0,1,2,\ldots,9$ we obtain $(n,m) = (12,10),(23,19),\ldots,(100,82)$ for a total of 10 ordered pairs. The answer is 19.