MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2007 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. In base 10, 14 + 44 = 58. $58 = 322_4$

2. If the sum of two primes is prime, then one of the primes must be 2. We have 2 + 3 = 5, 2 + 5 = 7, 2 + 11 = 13, 2 + 17 = 19, 2 + 29 = 31, and 2 + 41 = 43. This makes 6 pairs.

3. $(10A+B)(10B+C)-(100A+10B+C)=100AB-100A+10B^2-10B+10AC+BC-C$ $= 100A(B-1)+10B(B-1)+C(B-1)+10AC = (100A+10B+C)(B-1)+10AC$. Now, if $A = 9$, $B = 8$, and $C = 7$ we have $987\cdot 7+10\cdot 9\cdot 7 = 7539$. If $A = 8$, $B = 9$, and $C = 7$, we have $897\cdot 8+10\cdot 8\cdot 7 = 7736$. Hence, $ABC = 897$.

Round 2 Algebra 1

1. $a^2-6=5a+8$, $a^2-5a-14=0$. The prime number solution is 7.

2. Multiply the top by b and the bottom by a and obtain

$$\begin{aligned} abx + y &= ab \\ abx - y &= ab \end{aligned}$$

Subtracting gives $y = 0$ and $x = 1$.

Ans: (1, 0).

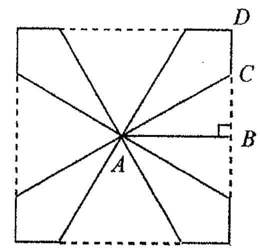
3. $x+\frac{9}{x}+y+\frac{25}{y} = 4 \rightarrow \frac{x^2+9}{x}+\frac{y^2+25}{y} = 4 \rightarrow$

$\frac{x^2+9}{x}+\frac{6x}{x}+\frac{y^2+25}{y}-\frac{10x}{x}-6+10 = 4 \rightarrow \frac{(x+3)^2}{x}+\frac{(y-5)^2}{y} = 0$. The only solution is (–3, 5), so 4(–3) + 7(5) = 23.

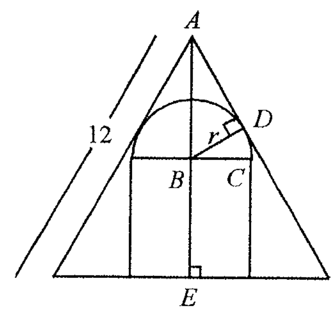
Round 3 – Geometry

1. The square is 13" on a side which is also the diameter of the circle. Ans. 13π

2. Let $BC = x$. Since ΔABC is a 30-60-90 right triangle, then $AB = x\sqrt{3}$ and $AC = 2x$. Thus, the side of the square is $2x\sqrt{3}$, making the perimeter equal to $8x\sqrt{3}$. The sum of AC and CD equals $2x + \left(x\sqrt{3} - x\right) = x + x\sqrt{3}$. Eight such sums make up the perimeter of the figure so its perimeter is $8x\sqrt{3} + 8x$. The ratio of perimeters equals $\dfrac{8x\sqrt{3} + 8x}{8x\sqrt{3}} = \dfrac{3 + \sqrt{3}}{3}$.



3. Draw the radius $\overline{BD}$ perpendicular to the side of the equilateral triangle. Since ΔABD is a 30-60-90 triangle, $AB = 2r$, and since $BC = r$, then the side of the square is $2r$. Thus, $AE = 4r = 6\sqrt{3} \rightarrow r = \dfrac{3\sqrt{3}}{2}$.



Round 4 – Algebra 2

1. The real solutions are $x \le -2$ or $-3 < x \le 5$. The positive integers and 4 and 5.

2. By the Arithmetic-Geometric Mean Inequality, $4^{x^2} + 4^{(x-1)^2} \ge 2\sqrt{4^{x^2} \cdot 4^{(x-1)^2}} = 2\sqrt{4^{2x^2-2x+1}}$. The minimum value of $2x^2 - 2x + 1$ occurs at $x = -\dfrac{-2}{4} = \dfrac{1}{2}$ and the min equals $2\left(\dfrac{1}{2}\right)^2 - 2\left(\dfrac{1}{2}\right) + 1 = \dfrac{1}{2}$. Hence, $4^{x^2} + 4^{(x-1)^2} \ge 2\sqrt{4^{1/2}} = 2^{3/2}$. The minimum of $\log_2\left(4^{x^2} + 4^{(x-1)^2}\right)$ is $\log_2 2^{3/2} = 3/2$. Range: $y \ge \dfrac{3}{2}$.

3. The equation $\left(\frac{k(k+1)}{2}\right)^2 - \frac{k(k+1)}{2} = 1980$ is quadratic in $\frac{k(k+1)}{2}$. Let $x = \frac{k(k+1)}{2}$ giving $x^2 - x - 1980 = 0 \rightarrow (x-45)(x+44) = 0$. Thus, $x = 45$, making $\frac{k(k+1)}{2} = 45$ and so $k = \boxed{9}$.

Round 5 – Analytic Geometry

1. $m\overline{FG} = \frac{4}{3}, m_L = -\frac{3}{4}$, $P(1,11)$, $y - 11 = -\frac{3}{4}(x-1)$

2 Let $f(x) = mx + b$, then $f^{-1}(x) = \frac{x}{m} - \frac{b}{m}$ making $m + \frac{1}{m} = \frac{17}{4} \rightarrow 4m^2 - 17m + 4 = 0$ $\rightarrow (4m-1)(m-4) = 0 \rightarrow m = \frac{1}{4}$ or 4. Now $b - \frac{b}{m} = 2$ so if $m = \frac{1}{4}$ then $b - 4b = 2$ $\rightarrow b = -\frac{2}{3}$. Since the equation $f(x) = \frac{x}{4} - \frac{2}{3}$ has a positive x-intercept, it satisfies the problem and the answer is $-\frac{2}{3}$. For $m = 4$, the equation is $f(x) = 4x + \frac{8}{3}$, but its x-intercept is negative.

3. Given $V(0,k)$ and $A(x, k - x^2)$, then $AV = \sqrt{x^2 + \left(k - x^2 - k\right)^2}$; $x^4 + x^2 = 30$. Then $\left(x^2 + 6\right)\left(x^2 - 5\right) = 0 \rightarrow x^2 = 5$. Since x is the radius, the area is 5π.

Round 6 – Trig and Complex Numbers

1. $\cos A = \frac{-2\sqrt{10}}{7}$, $\tan A = -\frac{3}{2\sqrt{10}} = -\frac{3\sqrt{10}}{20}$, $\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = -\frac{12\sqrt{10}}{31}$

2. $x^2 + x + 1 = 0 \rightarrow x^2 + 1 = -x \rightarrow x + \frac{1}{x} = -1$. Thus, $\left(x + \frac{1}{x}\right)^1 = -1$. Also, $\left(x + \frac{1}{x}\right)^2 = (-1)^2 \rightarrow x^2 + 2 + \frac{1}{x^2} = 1 \rightarrow x^2 + \frac{1}{x^2} = -1$. Thus, $\left(x^2 + \frac{1}{x^2}\right)^3 = -1$. If we obtain $x^4 + \frac{1}{x^4}$ in a similar fashion by squaring $x^2 + \frac{1}{x^2}$ we will again obtain -1. Thus, each expression represents -1 to an odd power so the sum equals $\boxed{-11}$.

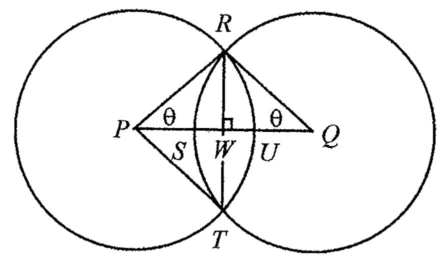
3. Using ΔABC with $m\angle ABC = \theta$, we have $\left(\frac{1}{\sqrt{2}}\right)^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cos\theta \rightarrow \cos\theta = \frac{3}{4}$.

Since $\cos 2\theta = 2\cos^2\theta - 1$, then $\cos\angle MNP = 2\left(\frac{3}{4}\right)^2 - 1 = \frac{1}{8}$. Then

$MP^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cdot \frac{1}{8} \rightarrow MP = \frac{\sqrt{7}}{2}$.

Team Round

1. Since $PR = RQ = 2$, then $SU = 1$, making $SW = 1/2 \rightarrow \cos\theta = 3/4$. Thus, the area of the sector RPU is $\frac{\cos^{-1}(3/4)}{2\pi}\cdot\pi\cdot 2^2 = 2\cos^{-1}(3/4)$. Since $RW = \sqrt{7}/2$, the area of



$\Delta RPW = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{\sqrt{7}}{2}$. Thus, the area of the region bounded by $\overline{RW}$, $\overline{WU}$, and $\overset{\frown}{RU}$ is $2\cos^{-1}\frac{3}{4} - \frac{3\sqrt{7}}{8}$. The area of the desired region is $4\cos^{-1}\frac{3}{4} - \frac{3\sqrt{7}}{4}$. The desired product is $4\cdot\frac{3}{4}\cdot\frac{3\sqrt{7}}{4} = \frac{9\sqrt{7}}{4}$.

2. The period is 8 and since $\frac{2007}{2} \div 8 = 125 + \frac{7}{16}$, $f\left(\frac{2007}{2}\right) = f\left(8\cdot\frac{7}{16}\right) = f\left(\frac{7}{2}\right)$. To obtain the y-value, determine the value of y when $x = 3.5$ using the equation $(x-4)^2 + (y-2)^2 = 4$. Thus, $(3.5-4)^2 + (y-2)^2 = 4 \rightarrow y = \frac{4+\sqrt{15}}{2}$.

3. $7x + 6y + 2z - 2(x - y + z) = 272 - 2\cdot 16 \rightarrow 5x + 8y = 240 \rightarrow x = 48 - \frac{8}{5}y$. Let $y = 5k$ for k an integer, then $x = 48 - 8k$, and substituting for both x and y into $x - y + z = 16$ we obtain $z = 13k - 32$. For x to be positive $k \le 5$ and for z to be positive $k \ge 3$. Thus, there are three lattice points of intersection lying in the first octant. For $k = 3$ we have (24, 15, 7), for $k = 4$ we have (16, 20, 20), and for $k = 5$ we have (8, 25, 33).

4. First note that $\{1, 1, 1\}$ satisfies the conditions. Next, given $\{a, 2a, a^2\}$, the first two and the first and third satisfy the conditions. It remains to consider the second and third. There are 4 cases:

i) $a^2 = 2(2a) \rightarrow a = 0, 4$. 0 fails so we have $a = 4$ giving $\{4, 8, 16\}$.
ii) $a^2 = (2a)^2 \rightarrow a = 0$
iii) $2a = (a^2)^2 \rightarrow a^3 = 2 \rightarrow a = 2^{1/3} \rightarrow \{2^{1/3}, 2^{4/3}, 2^{2/3}\}$
iv) $2a = 2(a^2) \rightarrow a = a^2 \rightarrow a = 0, 1$. From $a = 1$ we have $\{1, 2, 1\}$.

Had we considered $\{a, a^2, a^4\}$ and set $a = 2a^4 \rightarrow a^3 = 2^{-1}$, making $a = 2^{-1/3}$, giving the fifth solution $\{2^{-1/3}, 2^{-2/3}, 2^{-4/3}\}$.

5. $(x+4)^2 + y^2 + (x-4)^2 + y^2 = k \rightarrow 2x^2 + 2y^2 = k - 32$. Thus, $x^2 + y^2 = \dfrac{k-32}{2}$. If the circle has area of 9, then its radius is $\dfrac{3}{\sqrt{\pi}}$ so $\dfrac{k-32}{2} = \dfrac{9}{\pi} \rightarrow k = \dfrac{18}{\pi} + 32 = \dfrac{18 + 32\pi}{\pi}$.

6. Given $y = -\dfrac{x}{2} + 4 \leftrightarrow x + 2y - 8 = 0$, the distance from P to the line is $\dfrac{|6 + 2 \cdot 6 - 8|}{\sqrt{1^2 + 2^2}} =$ $2\sqrt{5}$. The minimum area will occur when the base $\overline{AB}$ is the least and that occurs when APB is an isosceles triangle. Then the altitude from P bisects $\angle APB$ and if x is half the base, then

$\tan(22.5) = \dfrac{x}{2\sqrt{5}}$. Since $\tan 45° = 1 = \dfrac{2\tan(22.5)}{1 - \tan^2(22.5)}$, then $1 = \dfrac{\frac{x}{\sqrt{5}}}{1 - \frac{x^2}{20}}$ giving

$x^2 + 4\sqrt{5} - 20 = 0 \rightarrow x = 2\sqrt{10} - 2\sqrt{5}$. The area is $x(2\sqrt{5}) = \boxed{20\sqrt{2} - 20}$.