CSAML 2019

April 4, 2019

Solutions

Round 1: Arithmetic and Number Theory

1. (1 point) Find the units digit in the number represented by 2019^{2019}.

Solution: The last digit is either 1 or 9. The last digit is 1 if the exponent is even and 9 if the exponent is odd.

Answer: 9

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2. (2 points) If $2^x \cdot 5^y \cdot 3^z = 28,800,000$, evaluate $x+y+z$.

Solution: $28,800,000 = 2^{10} \cdot 3^2 \cdot 5^5 \rightarrow x+y+z=17$

Answer 17.

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3. (3 points) For what value of n does $1!\cdot 1 + 2!\cdot 2 + 3!\cdot 3 + \cdots + n!\cdot n = 2019! - 1$?

Solution: Evaluate a few of the terms on the LHS of the equation to see what happens: $1!\cdot 1 + 2!\cdot 2 + 3!\cdot 3 = 1 + 4 + 18 = 23 = 4! - 1$.
Now try a few more terms: $1!\cdot 1 + 2!\cdot 2 + 3!\cdot 3 + 4!\cdot 4 = 23 + 96 = 119 = 5! - 1$. Notice a pattern, the last term is 1(one) less than the factorial on the RHS. So, $n = 2018$.

Answer: 2018

Round II: Algebra I, (Real numbers and no transcendental functions)

1.(1 point) Solve for all real values of y that satisfy the condition

$$6y+7 > 8y+2 > 4y+1.$$

Solution:

$$6y+7 > 8y+2 > 4y+1 \rightarrow 2y+6 > 4y+1 > 0 \rightarrow y > \frac{-1}{4}$$

$$5 > 2y \rightarrow y < \frac{5}{2} \quad \Rightarrow \frac{-1}{4} < y < \frac{5}{2}$$

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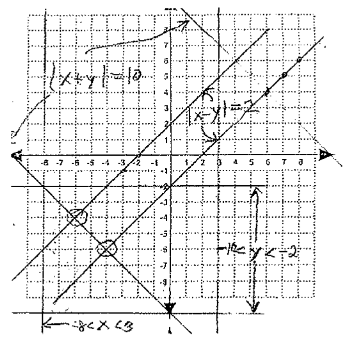
2. (2 points) Find all ordered pairs of integers (x, y) that satisfy all 4 conditions:
1) $-8 < x < 3$
2) $-10 < y < -2$
3) $|x - y| = 2$
4) $|x + y| = 10$

Solutions



Answer (-6, -4) , (-4, -6)

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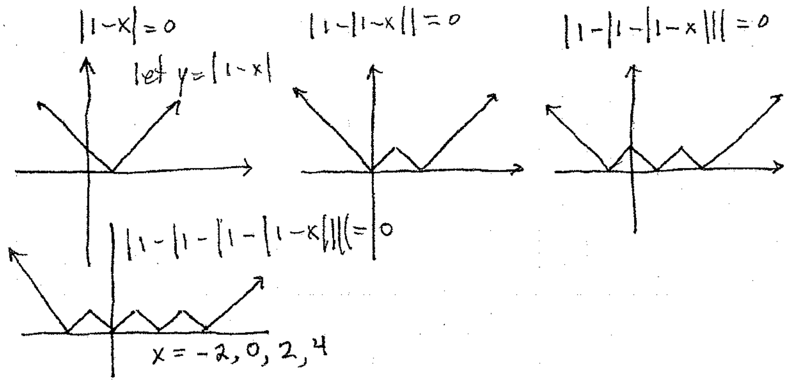
3. (3 points). Find all values of x that satisfy $\left|1-\left|1-\left|1-\left|1-x\right|\right|\right|\right| = 0$.

Answer; -2, 0, 2, 4 **Solution:** Look at this one unit of the problem at a time:
$|1-x| = 0 \rightarrow x = 1$

Now do the next set: $\left|1-\left|1-x\right|\right| = 0 \rightarrow 1 = |1-x| \rightarrow 1 - x = \pm 1 \rightarrow x = 0, 2$. Now 2 more sets. Here is a set of graphs to illustrate all of the solutions:

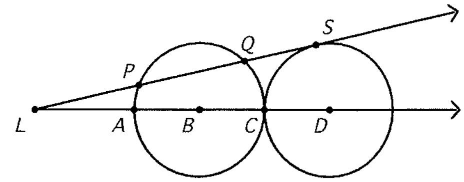


Round III: Geometry (figures are not to scale)

1. (1 point). Trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$, and $m\angle B = 112^\circ$ and $m\angle BCA = 33^\circ$. Line segments $\overline{AC}$ and $\overline{AD}$ have equal length. Find $m\angle DAC$.

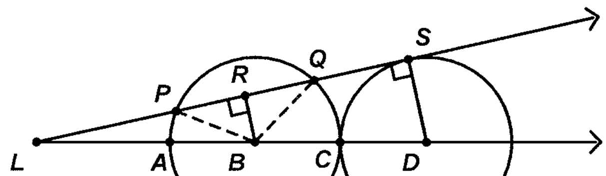
Solution: Since AB is parallel to CD, angles ABC and BCD are complimentary. Hence angle BCD is 180 – 112 = 68°. So angle ACD is 68 – 33 = 35°. Since we are given that AD = AC, triangle ACD is isosceles. Hence angle ADC = angle ACD = 35°. So angle DAC = 180 – 2(35) = 110°.
Answer: 110

2. (2 points) The diagram shows circles ω_B and ω_D externally tangent at C. The centers of the circles are B and D respectively, and each circle has radius 1. Ray $\overrightarrow{LD}$ passes through B and its other intersection point with ω_B is A. Ray $\overrightarrow{LS}$ is tangent to ω_D at S and intersects ω_B at P and Q. If $LA = 1.5$, find the length of PQ..



Solution: Draw lines perpendicular to LX through B and D, meeting LX at R and S, respectively. Note that LB = 2.5, LD = 4.5, and DS = 1. Using the similarity of triangles LBR and LDS, BR = (2.5/4.5)(1) = 5/9. Also, BP = 1 and BRP is a right triangle, so

$$PR = \sqrt{1^2 - (5/9)^2} = \frac{2\sqrt{14}}{9} \rightarrow PQ = 2\left(\frac{2\sqrt{14}}{9}\right) = \frac{4\sqrt{14}}{9}.$$



Answer: $\frac{4\sqrt{14}}{9}$

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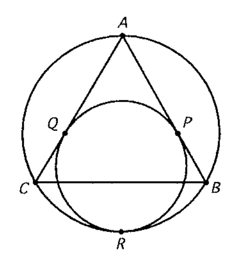
3. (3 points) Equilateral ΔABC of side length 6 is inscribed in circle ω. A smaller circle is internally tangent to ω at R and is tangent to $\overline{AB}$ and $\overline{AC}$ at P and Q, respectively. Find the distance PQ.

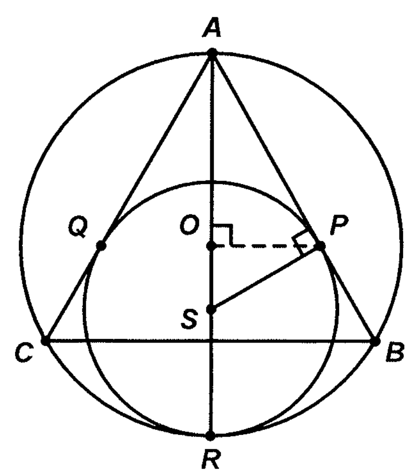


Solution: First consider only triangle ABC and circle ω. Let the center of ω be O. Draw a line segment through O perpendicular to BC, meeting BC at T. Then $m\angle OAB$ is 30° and BT = 3. So the radius of ω is OA = 3/(cos 30°) = 2√3. Now let the center of the smaller circle be S and note that SPA = 90° and $m\angle SAP = 30°$. So, letting the center of the smaller circle be *r*, AS = *r*/(sin 30°) = 2*r*. Remember that the point of tangency of the two circles be S. Then SR = *r*.

So AR = *r* + 2*r* = 3*r* is the diameter of ω, which is 4√3. So *r* = 4(√3)/3. Note that angle SPQ = 30°.

So $PQ = 2r\cos 30° = \frac{8\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{2} = 4$.

Answer: 4



Round IV: Algebra II

1. (1 point) Solve for $x \in \mathbb{R}$: $\log_4\left(\log_{\frac{1}{4}}(x)\right) = \frac{1}{2}$.

Solution: Convert the problem to exponential form: $\log_{\frac{1}{4}}(x) = 4^{\frac{1}{2}} = 2$.

Convert again, $x = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$ $x = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$ **Answer:** $\frac{1}{16}$

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2. (2 points) Solve for $x \in \mathbb{R}$: $x^{\frac{3}{5}} + 2x^{\frac{2}{5}} = 9x^{\frac{1}{5}} + 18$.

Solution: Let $a = x^{\frac{1}{5}}$. Now substitute into the original equation:

$$a^3 + 2a^2 = 9a + 18 \rightarrow a^2(a+2) = 9(a+2)$$
$$\rightarrow (a^2 - 9)(a+2) = 0 \rightarrow a = \pm 3, -2$$

if $a = -3 \rightarrow x^{\frac{1}{5}} = -3 \rightarrow x = -243$

if $a = 3 \rightarrow x^{\frac{1}{5}} = 3 \rightarrow x = 243$

if $a = -2 \rightarrow x^{\frac{1}{5}} = -2 \rightarrow x = -32$

Answer: -243, -32, 243

3.(3 points) If a and b are positive numbers that satisfy $a + \frac{1}{b} = 9$ and $b + \frac{4}{a} = 1$ then what is the value of ab?

Solution: Try "multiply the two equations": $\left(a + \frac{1}{b}\right)\left(b + \frac{4}{a}\right) = 9 \cdot 1$

$$\rightarrow ab + 4 + 1 + \frac{4}{ab} = 9 \rightarrow ab - 4 + \frac{4}{ab} = 0$$

$$(ab)^2 - 4(ab) + 4 = 0 \rightarrow (ab) = 2$$

Answer: 2

Round V: Analytic Geometry

1. (1 point)If the distance between (1, 3) and $(a, 2a)$ is 1 unit, find all possible values of a.

Solution: Setting up the equation $(1-a)^2+(3-2a)^2=1$ leads to the equation $5a^2-14a+9=0$, which factors into $(5a-9)(a-1)=0$, so $a=1$ and $a=\frac{9}{5}$.

Answer: $1, \frac{9}{5}$

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2. (2 points) Find the possible values of p such that the equation $x^2+y^2-6x+py+57=0$ has exactly one solution (x, y).

Solution: Completing the square gives $(x-3)^2+\left(y+\frac{p}{2}\right)^2=-48+\frac{p^2}{4}$. For this equation to have only one point as a solution, $-48+\frac{p^2}{4}=0$, so $p^2=192 \rightarrow p=\pm 8\sqrt{3}$.

Answer: $8\sqrt{3}, -8\sqrt{3}$

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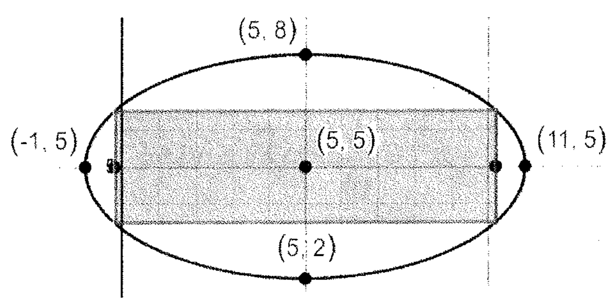
3. (3 points) A rectangle with sides parallel to the axes is inscribed in the ellipse with minor axis endpoints (5, 8) and (5, 2) and major axis endpoint (-1, 5). If each focus of the ellipse lies on a side of the rectangle, find the area of the rectangle.

Solution: See the diagram. The missing major axis endpoint has coordinates (11, 5). Knowing the center of the ellipse is (5, 5), major axis has length 12 and the minor axis has length 6, the equation of the ellipse is $\frac{(x-5)^2}{36}+\frac{(y-5)^2}{9}=1$. The distance c from the center to the $\sqrt{36-9}=3\sqrt{3}$ x-value of one focus is $\sqrt{36-9}=3\sqrt{3}$, so the length of the rectangle is $6\sqrt{3}$. The height of the rectangle can be found by letting $x=5+3\sqrt{3}$ and solving $\frac{3}{4}+\frac{(y-5)^2}{9}=1$.



leading to $(y-5)^2=\frac{9}{4}$,, so the y-value of a focus is a distance of $\frac{3}{2}$ from the major axis. This gives a height of the rectangle of 3, so the area is $18\sqrt{3}$. **Answer:** $18\sqrt{3}$

CSAML 2019

Solutions

Round VI: Trigonometry, Complex Numbers

1. (1 point) Toby walks 5 miles to the North and then 4 miles to the southwest. He is now d miles from his starting point. Find d^2.

Solution: We have a triangle with a 45° between sides 4 and 5. Thus, by the law of Cosines: $d^2 = 4^2 + 5^2 - 2(4)(5)\cos 45° \rightarrow d^2 = 16 + 25 - 40\left(\frac{\sqrt{2}}{2}\right) \rightarrow d^2 = 41 - 20\sqrt{2}$

Answer: $41 - 20\sqrt{2}$

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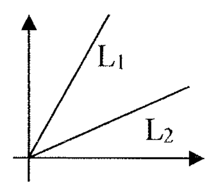
2. (2 points) Lines L_1 and L_2 both have positive slope. The slope of L_2 is four times that of L_1. The angle L_2 makes with the positive *x*-axis is twice the equivalent angle for L_1. Find the slope of L_1.

Solution: Let $m = \tan\theta$; $4m = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2m}{1-m^2}$

Now, $4m = \frac{2m}{1-m^2} \rightarrow 2 - 2m^2 = 1 \rightarrow 2m^2 = 1 \rightarrow m = \frac{\sqrt{2}}{2}$



Answer: $\frac{\sqrt{2}}{2}$

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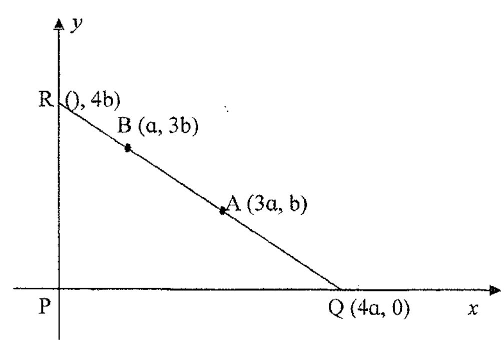
3. (3 points) PQR is a right triangle with right angle at P. Points A and B lie on QR with QA = RB = (1/4)QR. Furthermore, there is an acute angle θ with $\sin\theta$ = PA and $\cos\theta$ = PB. Find the length QR.

Solution:

$\sin\theta = PA = \sqrt{9a^2 + b^2}$, $\cos\theta = RB = \sqrt{a^2 + 9b^2}$

But, $\sin^2\theta + \cos^2\theta = 1 \rightarrow \left(9a^2 + b^2\right) + \left(a^2 + 9b^2\right) = 1$

$\rightarrow a^2 + b^2 = \frac{1}{10} \Rightarrow QR = \sqrt{16a^2 + 16b^2} = \sqrt{\frac{16}{10}} = \frac{2\sqrt{10}}{5}$



Answer: $\frac{2\sqrt{10}}{5}$

CSAML 2019

April 4, 2019 TEAM ROUND

1) For what positive integer value(s) of n is $n^2 - 132$ a perfect square?

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2) Let $\lfloor x \rfloor$ represent the greatest integer less than or equal to x and let $\{x\}$ represent the fractional part of x, $x - \lfloor x \rfloor$. Let $y = \dfrac{33\{\sqrt{2}\} + 62}{\{\sqrt{2}\} + 4}$. Find $\lfloor y \rfloor$.

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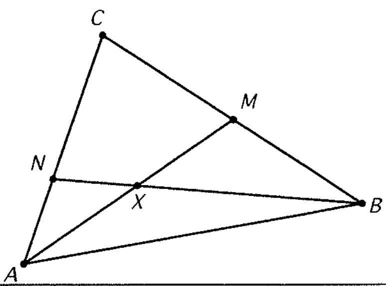
3) In ΔABC, M is the midpoint of side $\overline{BC}$ and point N divides side $\overline{AC}$ in a $2:3$ ratio. Line segments $\overline{AM}$ and $\overline{BN}$ intersect at X.

Compute the quantity $\dfrac{NX}{NB}$.



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4). If $x^4 + ax^2 + bx - 36 = 0$ has 4 distinct integer solutions for x, find all possible ordered pairs (a, b).

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5) Consider 2 circles in the xy-plane, one with equation $x^2 + y^2 - 1 = 0$ and the other with equation $x^2 + y^2 + x - \sqrt{3}y + k = 0$. The circles have centers C and M and intersect each other at points S and L. If $0 < k < 1$ and the area of quadrilateral $CSML$ is $\frac{1}{2}$, find the value of k.

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6) Note that, by De Moivre's theorem

$$\sum_{k=0}^{\infty} \frac{1}{2^k}(\cos\theta + i\sin\theta)^k = \sum_{k=0}^{\infty} \frac{1}{2^k}(\cos k\theta + i\sin k\theta).$$

Compute the sum $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k}\cos\frac{k\pi}{3}$.

Team Round Solutions

1) Solution: Let $m^2 = n^2 - 132$. Move the variables to one side and factor:

$$(n-m)(n+m) = 132 = 2\cdot 2\cdot 3\cdot 11$$

Note that $n+m > n-m$ and that both are even since their product is even. Thus there are two possibilities: $n+m = 2\cdot 3\cdot 11,\ n-m = 2$ OR $n+m = 2\cdot 11,\ n-m = 2\cdot 3$. These give us $n = 34, m = 32$ and $n = 14, m = 8$.

Answer: 14, 34

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2) **Solution:**

$$y = \frac{33\left\{\sqrt{2}\right\}+62}{\left\{\sqrt{2}\right\}+4} = \frac{33\left\{\sqrt{2}-1\right\}+62}{\sqrt{2}-1+4} = \frac{33\sqrt{2}+29}{\sqrt{2}+3}$$

$$= \frac{\left(33\sqrt{2}+29\right)\left(\sqrt{2}-3\right)}{2-9} = \frac{33\cdot 2 + 29\sqrt{2} - 99\sqrt{2} - 87}{-7}$$

$$= \frac{66 - 70\sqrt{2} - 87}{-7} = \frac{-21 - 70\sqrt{2}}{-7} = 3 + 10\sqrt{2}$$

Now find $\lfloor y \rfloor$. $\left\lfloor 3 + 10\sqrt{2} \right\rfloor = 3 + 14 = 17$

Answer: 17

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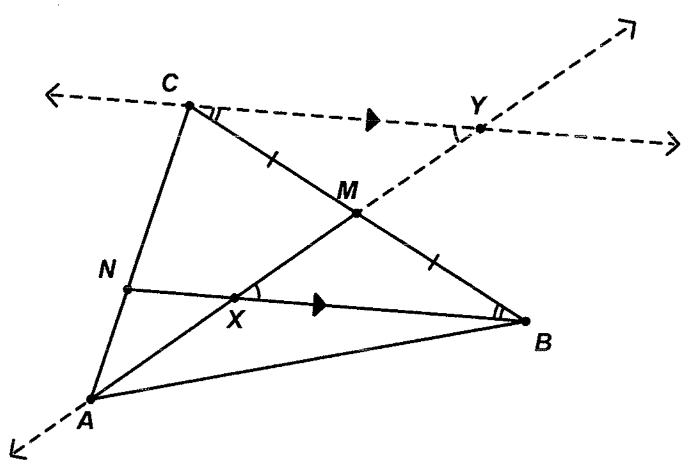
3) Solution 1, using similarity:

Construct a line through C, parallel to $\overline{NB}$.
Extend $\overline{AM}$ and let Y denote its intersection with the parallel line.

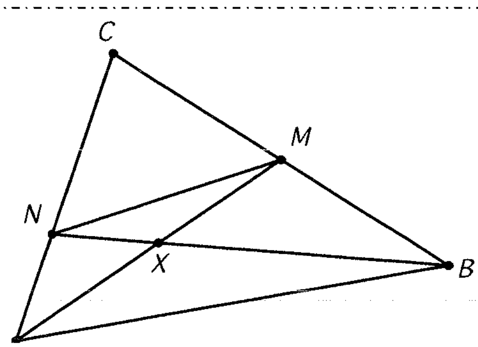


Then $\Delta XBM \cong \Delta YCM$ by AAS, and thus $\overline{XB} \cong \overline{YC}$.

Notice that $\Delta ANX \sim \Delta ACY$ by AA, with a similarity ratio of $2/5$. Therefore,

$$\frac{2}{5} = \frac{AN}{AC} = \frac{NX}{CY} = \frac{NX}{XB}, \text{and thus } \frac{NX}{NB} = \frac{NX}{NX+XB} = \frac{2}{7}.$$

Solution2, using areas:



Draw in $\overline{NM}$. We will use $[ABC]$ to denote the area of ΔABC.

We can assume that $[ABC] = 1$.

Solutions

Using the principle that the ratio of areas of triangles of the same height equals the ratio of their bases, we know that

$[CMA]=[MBA]=\frac{1}{2},\ [CNB]=\frac{3}{5},[NAB]=\frac{2}{5}.$

Furthermore, $[CNM]=\frac{3}{10},[NAM]=\frac{2}{10}=\frac{1}{5}.$

Then, $\frac{AX}{AM}=\frac{[AXN]}{[AMN]}=\frac{[AXB]}{[AMB]}$, so $\frac{[AXN]}{1/5}=\frac{[AXB]}{1/2}.$

Therefore, $\frac{NX}{XB}=\frac{[AXN]}{[AXB]}=\frac{2}{5}$, from which it follows that $\frac{NX}{NB}=\frac{NX}{NX+XB}=\frac{2}{7}.$

Answer: 2/7

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4) **Solution:**

The constant term is -36 which must be the product of the solutions. Since there is no cubic term, it follows that the sum of the solutions must be 0. There are two combinations of 4 distinct integers with those properties: -6, 3, 2, **1** and 6, -3, -2, **-1**. Setting up the 2 products $(x+6)(x-3)(x-2)(x-1)$ and $(x-6)(x+3)(x+2)(x+1)$ gives polynomials of $x^4-25x^2-60x-36$ and $x^4-25x^2+60x-36$, providing 2 ordered pairs, (-25, -60) and (-25, 60).

Answer: $(-25,-60),(-25,60)$

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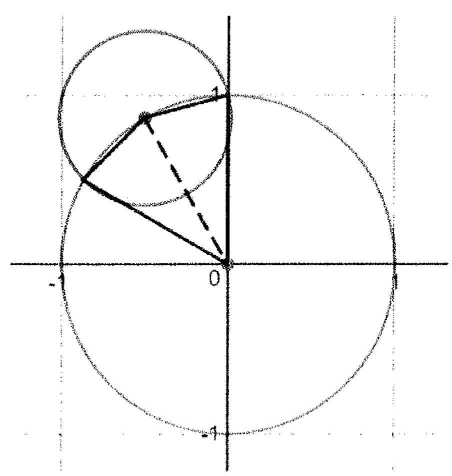
5) **Solution:**

See the diagram. The first circle is the unit circle. Completing the square for the equation of the second circle gives the equation $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{\sqrt{3}}{2}\right)^2=1-k$. If r is the radius of this circle, this means that $r^2=1-k$. Since the center of the second circle is $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, it lies on the circumference of the unit circle. Due to the restriction on k, the second circle must have a radius between 0 and 1, and is therefore smaller than the unit circle. The resulting quadrilateral is a kite which can be decomposed into two isosceles triangles with base r and legs of length 1 (the radii of the unit circle). The area of the quadrilateral is then



$2\left(\frac{1}{2}\right)r\sqrt{1-\frac{r^2}{4}}$, giving the equation $r\sqrt{1-\frac{r^2}{4}}=\frac{1}{2}$. Squaring both sides yields $r^2\left(1-\frac{r^2}{4}\right)=\frac{1}{4}$, which gives $r^4-4r^2+1=0$. Using the quadratic formula, we get $r^2=2\pm\sqrt{3}$, but since $r<1$, $r^2=2-\sqrt{3}$. Setting $1-k=2-\sqrt{3}$ gives $k=\sqrt{3}-1$.

Answer: $\sqrt{3}-1$

6) Solution:

$$\sum_{k=0}^{\infty}\frac{1}{2^k}\cos\frac{k\pi}{3}=\mathrm{Re}\left[\sum_{k=0}^{\infty}\frac{1}{2^k}\left(\cos\frac{k\pi}{3}+i\sin\frac{k\pi}{3}\right)\right]=\mathrm{Re}\left[\sum_{k=0}^{\infty}\frac{1}{2^k}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^k\right]$$

$$\sum_{k=0}^{\infty}\frac{1}{2^k}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^k=\frac{1}{1-\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}=\frac{1}{1-\frac{1}{2}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}$$

$$=\frac{1}{1-\frac{1}{4}-\frac{i\sqrt{3}}{4}}=\frac{4}{3-i\sqrt{3}}=\frac{4\left(3+i\sqrt{3}\right)}{9+3}=\frac{1}{3}\left(3+i\sqrt{3}\right)=1+\frac{i\sqrt{3}}{3}$$

So, $\sum_{k=0}^{\infty}\frac{1}{2^k}\cos\frac{k\pi}{3}=1$

Answer: 1

Short Answers:
CSAML 2019

Round 1

1) 9

2) 17

3) 2018

Round 2

1) $\frac{-1}{4} < y < \frac{5}{2}$

2) $(-6,-4),(-4,-6)$

3) -2, 0, 2, 4

Round 3

1) 110

2) $\frac{4\sqrt{14}}{9}$

3) 4

Round 4

1) $\frac{1}{16}$

2) -243, -32, 243

3) 2

Round 5

1) $1, \frac{9}{5}$

2) $8\sqrt{3}, -8\sqrt{3}$

3) $18\sqrt{3}$

Round 6

1) $41-20\sqrt{2}$

2) $\frac{\sqrt{2}}{2}$

3) $\frac{2\sqrt{10}}{5}$

TEAM

1) 14, 34

2) 17

3) $\frac{2}{7}$

4) $(-25,-60),(-25,60)$

5) $\sqrt{3}-1$

6) 1