MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS - 2017 - SOLUTIONS

Round 1: Arithmetic and Number Theory

1. The two-digit reversible primes are 11, 13, 17, 31, 37, 71, 73, 79, 97. There are **9** of them.

2. $p^3+6p^2=p^2(p+6)$. Thus, $p+6$ must be a perfect square for $0<p<100$. We can't have $p+6$ equal to either 1 or 4, since that makes p negative. $p+6$ can't be larger than 100, since otherwise, p is greater than 100. Thus, $p+6=9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,$ and 100. There are **8** values of p that work.

3. Note that $4+4^2+4^3+4^4=340$. Since $4^{n+1}+4^{n+2}+4^{n+3}+4^{n+4}=4^n\left(4+4^2+4^3+4^4\right)=$ 340, then groups of 4 consecutive terms have a sum divisible by 17. Since 2016 = 4(504), then the sum $4+4^2+4^3+\ldots+4^{2016}$ is divisible by 17. When $1+4+4^2+4^3+\ldots+4^{2016}$ is divided by 17, there is a remainder of **1**.

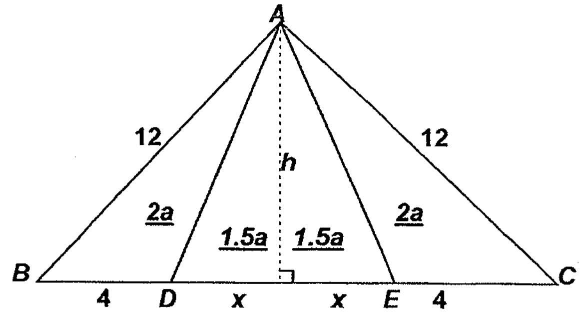
Round 2 Algebra 1

1. The discriminant must be positive and equal to a perfect square. Thus, $20^2-4\cdot7\cdot k=n^2$ meaning that $400-28k>0\to k<14\frac{2}{7}$. For $k=14,\ 13,$ and 12, respectively, the discriminant equals 8, 36, and 64. The answer is **13**.

2. $$\frac{4}{y+3}-2-\frac{2y}{3-y}+\frac{12y}{9-y^2}\Leftrightarrow\frac{4}{y+3}-\frac{2}{1}+\frac{2y}{y-3}-\frac{12y}{(y+3)(y-3)}$$
Combining terms, $$\frac{4(y-3)-2(y^2-9)+2y(y+3)-12y}{(y+3)(y-3)}=\frac{-2y+6}{(y+3)(y-3)}$$
$$=\frac{-2\cancel{(y-3)}}{(y+3)\cancel{(y-3)}}=-\frac{\mathbf{2}}{\mathbf{y+3}}.$$

3. $(By+Gx)=(Bx+Gy)+10\Rightarrow(By+Gx)-(Bx+Gy)=10$
$\Rightarrow B(y-x)-G(y-x)=10\Leftrightarrow(B-G)(y-x)=10$
(Note: $B>G\Rightarrow y>x$ is consistent with the hamburger costing more than the roll.)
But more importantly, $B-G$ can only be a factor of 10, namely, **1, 2, 5,** and **10**.

Round 3 – Geometry



1. $\text{area}(\Delta ACE)=\frac{1}{2}\cdot h\cdot 4=2h$

$\text{area}(\Delta ADE)=2\left(\frac{1}{2}\cdot h\cdot x\right)=hx$

$\frac{2h}{hx}=\frac{4}{5}\Rightarrow x=2.5\Rightarrow BC=2(4+2.5)=\underline{\mathbf{13}}$.

Alternately, since $\text{area}(\Delta ACE):\text{area}(\Delta ADE)=4:5$ and these triangles have the same altitude from vertex A, their bases must be in a 4 : 5 ratio, implying $DE = 5$ and $BC = \underline{\mathbf{13}}$.

2. From the three tangent lines, we see that the diameter of C is 2 + 8 = 10, so its radius is 5. If the center were at (–3, –8). The line $y=\frac{1}{3}x+\frac{14}{3}$ would not intersect circle C at all. Thus, the center is (-3, 2). The equation of the circle is $(x+3)^2+(y-2)^2=25$. The equation of the line can be written as $x-3y=-14$. Substituting for x, we get $(3y-11)^2+(y-2)^2=25$ or $9y^2-66y+121+y^2-4y+4-25=0$. Collecting terms we get $10y^2-70y+100=0$ or $y^2-7y+10=0$. Then $y = 2$ or $y = 5$. Substituting we get $x = -8$ or $x = 1$. The points of intersection are **(-8, 2)** and **(1, 5)**.

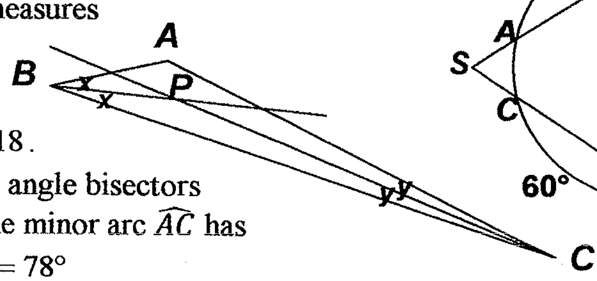
3. The number of diagonals is given by

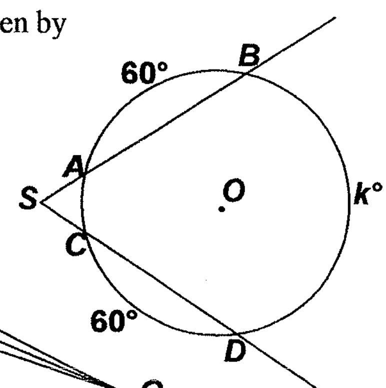
$d=\frac{n(n-3)}{2}\Rightarrow(n,d)=(4,2),(5,5),(6,9),(7,14),(8,20),(9,27),\underline{(10,35)}$ - Bingo!

In a regular polygon, the angle measure of an interior angle is given by

$\frac{180(n-2)}{n}\Rightarrow\frac{180(10-2)}{10}=18\cdot 8=144$

Let the other two angles have measures $(2x)^\circ$ and $(2y)^\circ$. Then:





$2x+2y+144=180\Rightarrow x+y=18$.

The obtuse angle formed by the angle bisectors measures $180^\circ-18^\circ=162^\circ$. The minor arc $\widehat{AC}$ has measure $360^\circ-(162^\circ+2\cdot 60^\circ)=78^\circ$

Therefore, $m\angle S=\frac{1}{2}(162-78)=\underline{\mathbf{42}}^\circ$.

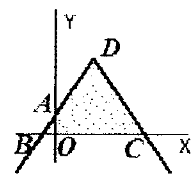
Round 4 – Algebra 2

1. $x^2 4^x - 2^{2x+2} - 64x^2 + 256 = x^2 4^x - 4^x 4 - 64(x^2-4) = (2^x)^2(x^2-4) - 64(x^2-4) =$
$\left[(2^x)^2 - 64\right]\cdot(x^2-4) = \underline{\mathbf{(2^x+8)(2^x-8)(x+2)(x-2)}}$.

2. $S_6 = \frac{6}{2}(2a+5d) = 261$ and $S_9 = \frac{9}{2}(2a+8d) = 297 \Rightarrow \begin{cases} 2a+5d=87 \\ 2a+8d=66 \end{cases} \Rightarrow 3d = -21 \Rightarrow d = -7$.
$\Rightarrow (a, d) = \underline{\mathbf{(61, -7)}}$.

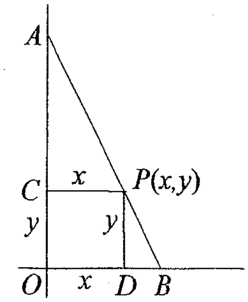
3. $P(0) = 100\cdot\frac{3+4^0}{1+2^0} = 200$. We want t such that $P(t) = 100\frac{3+4^t}{1+2^t} = 400 \rightarrow \frac{3+4^t}{1+2^t} = 4 \rightarrow$
$3+4^t = 4+4\cdot 2^t \rightarrow \left(2^t\right)^2 - 4\cdot 2^t - 1 = 0$. Then $2^t = \frac{4\pm\sqrt{16--4\cdot 1}}{2} = 2\pm\sqrt{5}$. Choose
$2^t = 2+\sqrt{5}$ making $t = \log_2\left(2+\sqrt{5}\right)$. Thus, $K = \underline{\mathbf{2+\sqrt{5}}}$.

Round 5 – Analytic Geometry



1. The graph is an inverted V with a vertex at $D\left(\frac{5}{2}, 6\right)$. It intersects the y-axis at $A(0,1)$ and the x-axis at $C\left(\frac{11}{2}, 0\right)$. Let $B = \left(\frac{5}{2}, 0\right)$ and let O be the origin. The region bounded by the function and the axes breaks up into trapezoid $ADBO$ and right triangle DBC. The area of $ADBO$ is $\frac{1}{2}\cdot\frac{5}{2}(1+6) = \frac{35}{4}$.
The area of DBC is $\frac{1}{2}\cdot\left(\frac{11}{2}-\frac{5}{2}\right)\cdot 6 = 9$. The total is $\underline{\mathbf{\frac{71}{4}}}$.

Or: Let $E\left(-\frac{1}{2}, 0\right)$ be the left hand x-intercept. The area of the region bounded by the function and the axes is the difference between the areas of ΔEDC and ΔEOA which is $\frac{1}{2}\left(\frac{11}{2} - -\frac{1}{2}\right)6 - \frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \underline{\mathbf{\frac{71}{4}}}$.



2. In the diagram, $AP = 12$ and $PB = 6$, so $AC = \sqrt{144 - x^2}$.

Since $\Delta ACP \square \Delta PDB$, $\frac{\sqrt{144-x^2}}{y} = \frac{12}{6} \to 144 - x^2 = 4y^2$.

When P is the vertex of a square, $x = y$, giving $144 = 5x^2$

$\to x^2 = \frac{144}{5}$ and that is the area of the square. Ans: $\underline{\mathbf{\frac{144}{5}}}$

3. If the slope is m, then from $y - 8 = m(x-4)$ we obtain $B = \left(\frac{4m-8}{m}, 0\right)$. If the x-coordinate of B is greater than 4, B lies to the right of A and the slope is negative. Then $\frac{1}{2} \cdot \left(\frac{4m-8}{m}\right) \cdot 8 = 20 \to m = -8$. If the x-coordinate of B is less than 4, B lies to the left of A and the slope is positive, making $\frac{4m-8}{m}$ negative. Then $-\frac{1}{2} \cdot \left(\frac{4m-8}{m}\right) \cdot 8 = 20 \to m = \frac{8}{9}$.

The slopes are $\underline{\mathbf{-8} \text{ and } \mathbf{\frac{8}{9}}}$.

Round 6 – Trig and Complex Numbers

1. $\csc\alpha = \frac{25}{24}$, where $90° < \alpha < 180° \Rightarrow \sin\alpha = \frac{24}{25}, \cos\alpha = -\frac{7}{25}$.

$\cot\beta = \frac{12}{5}$, where $180° < \beta < 270° \Rightarrow \sin\beta = -\frac{5}{13}$ and $\cos\beta = -\frac{12}{13}$.

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(-\frac{7}{25}\right)\left(-\frac{12}{13}\right) - \left(\frac{24}{25}\right)\left(-\frac{5}{13}\right) = \frac{84+120}{325} = \underline{\mathbf{\frac{204}{325}}}$.

2. If $z=1+2i$, then $\overline{z}=1-2i$, and $\frac{1}{z}=\frac{1}{1+2i}\cdot\frac{1-2i}{1-2i}=\frac{1-2i}{5}$. In the rectangular coordinate plane, we have the vertices at $A(1,2)$, $B(1,-2)$, and $C\left(\frac{1}{5},-\frac{2}{5}\right)$. Let $\overline{AB}$ be the base; here $AB=4$. The distance from C to $\overline{AB}$ is $1-\frac{1}{5}=\frac{4}{5}$ so the area of ΔABC is $\frac{1}{2}\cdot\frac{4}{5}\cdot 4=$ $\underline{\mathbf{\frac{8}{5}}}$.

3. $\tan\theta=\dfrac{\frac{1}{3}-\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right)}=\dfrac{-\frac{1}{6}}{\frac{7}{6}}=-\frac{1}{7}$. θ must be in quadrant 4 $\therefore$ $\sin\theta=\frac{-1}{\sqrt{50}}$ and $\cos\theta=\frac{7}{\sqrt{50}}$.

Then $\sin 2\theta=-\frac{14}{50}=$ $\underline{\mathbf{-\frac{7}{25}}}$

Team Round

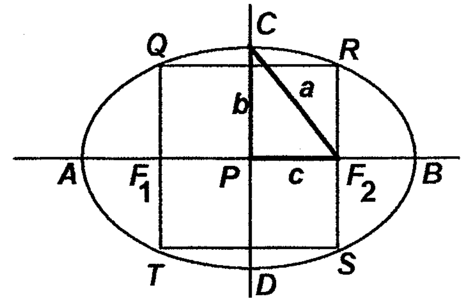
1. Considering only prime values of $n + 1$,
$n+1=2 \Rightarrow 9(1)^4 + 121(2)^2 = 493 = P^2$ (rejected, since $19^2 = 361$ is too small, $23^2 = 529$ is too large and there are no other primes between 19 and 23)
$n+1=3 \Rightarrow 9(2)^4 + 121(3)^2 = 144 + 1089 = 1233 = P^2$ (rejected, $31^2 = 961$ and $37^2 = 1369$)
$n+1=5 \Rightarrow 9(4)^4 + 121(5)^2 = 2304 + 3025 = 5329$
Since the squares of 70 and 75 are easily computed ($70^2 = 4900$ and $75^2 = 5625$) – see below, we have trapped the required value. The only possible primes are 71 and 73.
Only 73^2 ends in 9 and, multiplying out, we get exactly 5239. Thus, $P =$ **<u>73</u>**.

<u>FYI #1</u>: How was 75^2 **<u>easily</u>** determined to be 5625? Clearly, the rightmost two digits of the square of a number ending in 5 are always 25. Symbolically, $(\underline{x5})^2 = ...25$.
Not so clearly, the missing digits may be found by multiplying $x(x + 1)$. For example,
$25^2: 2(3) = 6 \Rightarrow 625$
$35^2: 3(4) = 12 \Rightarrow 1225$
$45^2: 4(5) = 20 \Rightarrow 2025$
...
$105^2: 10(11) = 110 \Rightarrow 11025$

Why does this work?
The number ending in 5, namely $\underline{x5}$, can be represented algebraically as $10x + 5$.
Squaring,
$$(10x+5)^2 = 100x^2 + 2(5)(10)x + 25 = 100x^2 + 100x + 25 = 100(x^2 + x) + 25 = 100\underline{x(x+1)} + 25$$
The factor of 100 simply shifts the product two places to the left and insures that the rightmost two digits are 25. A very useful shortcut indeed!
In fact, x may be greater than 9.
Check it out: $105^2 = 11025,\ 115^2 = 13225,\ 125^2 = 15625, ..., 205^2 = 42025, ...$.

2. Since $91 = 7 \cdot 13$, if m is a multiple of 7, then n is also; and, if m is a multiple of 13, then n is also. Thus, the rejected ordered pairs are $(7,84),...,(42,49),\ (13,78),(26,65),(39,52)$. There are 45 possible ordered pairs (m,n), where $m + n = 91$. Therefore,
$$S = (1+...+45) - 7(1+...+6) - 13(1+..+3) = \frac{45 \cdot 46}{2} - 7\frac{6 \cdot 7}{2} - 78$$
$45 \cdot 23 - 3 \cdot 49 - 78 = 3(15 \cdot 23 - 49 - 26) = 3(345 - 75) = 3(270) =$ **<u>810</u>**.

3. 8 can only be written as the sum of integer perfect squares in one way, namely 4 + 4. Thus, $(x-1)^2 = 4 \Rightarrow x = -1, 3$ or $(y-5)^2 = 4 \Rightarrow y = 3, 7$ and the square has vertices at $S(3, 3)$, $R(3, 7)$, $Q(-1, 7)$ and $T(-1, 3)$. The diagonals intersect at $P(1, 5)$. This is the center of the ellipse. Since $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ is the equation of a horizontal ellipse, the foci lie on the vertical sides of the square and $c = 2$. The vertical sides of the square are focal chords of the ellipse. Therefore, $\frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a$.



Since, for any ellipse, $a^2 = b^2 + c^2$, we have $a^2 - b^2 = 4$.

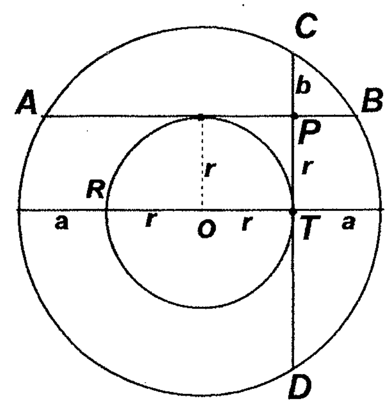
Substituting, $a^2 - 2a - 4 = 0 \Rightarrow \boxed{a = \frac{2+\sqrt{20}}{2} = 1+\sqrt{5}} \Rightarrow$

$a^2 = 6 + 2\sqrt{5}$ and $b^2 = 2 + 2\sqrt{5}$.

$(abc)^2 = a^2b^2c^2 =$

$(6+2\sqrt{5})(2+2\sqrt{5})(4) = 16(3+\sqrt{5})(1+\sqrt{5}) = 16(8+4\sqrt{5}) = 64(2+\sqrt{5}) \Rightarrow \underline{\mathbf{(64,2,5)}}$.

4. Applying the Product-Chord Theorem at point T,



$(r+b)^2 = a(2r+a)$

$\Rightarrow r^2 + 2br + b^2 = 2ar + a^2 \Leftrightarrow r^2 + 2(b-a)r + b^2 - a^2 = 0$

Solving for r, using the quadratic formula, we have

$$r = \frac{2(a-b) \pm \sqrt{4(b-a)^2 - 4(b^2-a^2)}}{2}$$

$\Rightarrow r = (a-b) \pm \sqrt{\cancel{b^2} - 2ab + a^2 \cancel{-b^2} + a^2} = \boxed{(a-b) \pm \sqrt{2a(a-b)}}$

For r to be rational, $2a(a-b)$ must be a perfect square x^2.

$a = 2 \Rightarrow b = 1$ - rejected $(a \neq 2b)$

$a = 3 \Rightarrow b = 1, 2 \Rightarrow x^2 = 6(2,1)$ - rejected

$a = 4 \Rightarrow b = 1, 3 \Rightarrow x^2 = 8(3,1)$ - rejected

$a = 5 \Rightarrow b = 1,2,3,4 \Rightarrow x^2 = 10(4,3,2,1)$ - rejected

$a = 6 \Rightarrow b = 1,2,4,5 \Rightarrow x^2 = 12(5,4,2,1)$ -rejected

$a = 7 \Rightarrow b = 1,2,3,4,5,6 \Rightarrow x^2 = 14(6,5,4,3,2,1)$ - rejected

$a = 8 \Rightarrow b = 1,2,3,5,6,7 \Rightarrow x^2 = 16(7,6,5,3,2,1) \Rightarrow x^2 = 16,\ r = (8-7) \pm \sqrt{16} \Rightarrow r = 5$

Thus, $(a,b,r) = \underline{\mathbf{(8,7,5)}}$.

5. At least 4 means 4, 5 or 6 heads. Consider the expansion of $(p + q)^6$, where q, the probability of tails, $= (1 - p)$. $(p+q)^6 = p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$. (Note the coefficients from the 6th row of Pascal's Triangle.) The first term represents the probability of exactly 6 heads. The second term represents the probability of exactly 5 heads, since there are 6 possible arrangements of 5H's and 1 T. $\left(\frac{6!}{5!1!}\right)$ - THHHHH, HTHHHH, …, HHHHHT

The third term represents the probability of exactly 4 heads, since there are 15 possible arrangements of 4H's and 2 T. $\left(\frac{6!}{4!2!}\right)$ - TTHHHH, THTHHH, THHTHH, THHHTH, THHHHT, …, HHHHTT

P($\geq$ 4 heads) $= p^6 + 6p^5q + 15p^4q^2 = p^6 + 6p^5(1-p) + 15p^4(1-p)^2 > 0.5$

$p^4(p^2 + 6p(1-p) + 15(1-p)^2 = p^4(10p^2 - 24p + 15) > 0.5$

Constructing a lookup table, we require $P > \frac{\boxed{5000}}{10000}$.

The following table summarizes the numerators for possible values of p.

p	P	Numerator
.1	$\frac{1}{10^4}\left(10\cdot 1.1^2 + .6\right)$	$12.7 < 5000$
.2	$\frac{2^4}{10^4}\left(10\cdot 1 + .6\right)$	$16(10.6) < 5000$
.3	$\frac{3^4}{10^4}\left(10\cdot .81 + .6\right)$	$81(8.7) < 5000$
.4	$\frac{4^4}{10^4}\left(10\cdot .64 + .6\right)$	$256(7) < 5000$
.5	$\frac{5^4}{10^4}\left(10\cdot .49 + .6\right)$	$625(5.5) < 5000$
.6	$\frac{6^4}{10^4}\left(10\cdot .36 + .6\right)$	$1296(4.2) > 5000$ (BINGO!)

Thus, to the nearest tenth, $p =$ **0.6**.
(In fact, the last two rows of the table would have been sufficient.)

6. Let $(\sin P, \sin Q, \sin R) = (2n, 3n, 4n)$. Invoking the Law of Sines, $\frac{\sin P}{p} = \frac{\sin Q}{q} = \frac{\sin R}{r}$, without loss of generality, we may let the sides of ΔPQR $(p,q,r) = (2,3,4)$. Invoking the Law of Cosines, we have

$$\begin{cases} 2^2 = 3^2 + 4^2 - 2\cdot 3\cdot 4\cos P \\ 3^2 = 2^2 + 4^2 - 2\cdot 2\cdot 4\cos Q \\ 4^2 = 2^2 + 3^2 - 2\cdot 2\cdot 3\cos R \end{cases} \Leftrightarrow \begin{cases} 4 = 25 - 24\cos P \\ 9 = 20 - 16\cos Q \\ 16 = 13 - 12\cos R \end{cases} \Leftrightarrow (\cos P, \cos Q, \cos R) = \left(\frac{7}{8}, \frac{11}{16}, -\frac{1}{4}\right).$$

Multiplying through by 16 to clear the fractions, the required ratio is 14 : 11: -4 and $(a,b,c) = \underline{\mathbf{(14,11,-4)}}$.