NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2009

Round 1: Arithmetic and Number Theory

1. _____

2. _____

3. _____

1. Determine the positive integral base that requires exactly six digits to express 2009_{10}.

2. Let p be a prime and set $N = p^3 + p$. If N has exactly 4 factors, determine the value of N.

3. Indistinguishable 1 by 1 green squares and indistinguishable 1 by 3 red rectangles are placed in a row forming a 1 by 10 rectangle. The 1 by 10 rectangle can be composed of all of one figure or some of both figures. How many different arrangements form the 1 by 10 rectangle?

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Round 2: Algebra 1

1. _____

2. _____

3. _____

1. For real x and y, if $x^2 - xy + y^2 = \dfrac{42}{x+y}$ and $x^2 + xy + y^2 = \dfrac{12}{x-y}$, what is the value of x?

2. Let p and d be positive integers. If $|x - p| < d$ has 7 integral solutions, determine the value of d.

3. The following five distinct integers are written in increasing order: 7, 8 10, x, y. List all ordered pairs (x, y) such that the average of the five numbers equals the median of them. (Proper notation must be used in your answer.)

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Round 3: Geometry

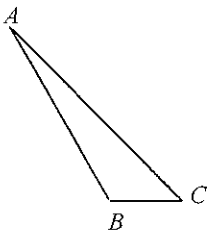
1. _____

2. _____

3. _____ : _____

1. In right triangle ABC, $m\angle C = 90$. D lies on $\overline{BC}$ such that $AD = DB = CD\sqrt{10}$. Determine the numerical value of $\frac{AC}{CD}$.

2. Find, in simplified exact form, the number of square units in the area of ΔABC given $AC = 6$, $m\angle B = 120°$, and $m\angle A = 15°$.



3. In equilateral triangle ABC, medians $\overline{CD}$ and $\overline{BE}$ intersect at F. Determine the ratio of the perimeter of $ADFE$ to the perimeter of FBC, in simplified, exact form.

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Round 4: Algebra 2

1. _____

2. _____

3. _____

1. A polynomial has integer coefficients and the leading coefficient is 1. If two of the roots are $2i$ and $1 - i$, find the sum of the coefficients of the polynomial of least degree.

2. Let f be a cubic whose coefficients are all ~~positive~~ non-negative integers and $f(1) = 3$. Find the sum of the least and greatest possible values of $f(2)$. (The coefficient of x^3 is not zero.)

3. Let $a_1 = 2$ and $a_n = a_{n-1} + 2$. For how many values of n, $1 \le n \le 2009$, is $\sum_{i=1}^{n} a_i$ divisible by 7?

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Round 5: Analytic Geometry

1. _____

2. _____

3. _____

1. Let f be a strictly increasing function on the open interval (1, 2). Determine the number of integral values of k such that $f(x) = x^3 + x^2 + 3x + k$ has a zero on (1, 2).

2. Determine all values of k such that the graph of $f(x) = \dfrac{x^3 + 5x}{2x^3 - x^2 + k}$ intersects its horizontal asymptote in 2 points.

3. The vertices of ΔABC are $A(1,2)$, $B(4,k)$, and $C(k,8)$.
   Find the sum of all values of k such that the area of ΔABC is 9.

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Round 6: Trig and Complex Numbers

1. _____

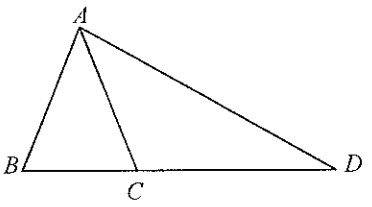
2. _____

3. _____

1. In ΔABC, $BA = 6$, $BC = 8$, and $\angle B$ is acute. If the area of ABC is 20 square units, find $\cos B$.

2. Determine the largest real number a such that $\left|\frac{1}{1+i} - a\right| = 1$.

3. Given $AB = AC = 8$, $BC = 4$, and $CD : AD = 3 : 4$, find the number of units in the length of $\overline{AD}$.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2009

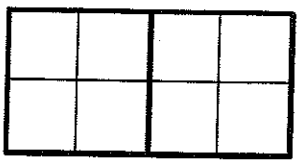
Team Round

1. _____ **4.** _____

2. _____ **5.** _____

3. _____ **6.** _____

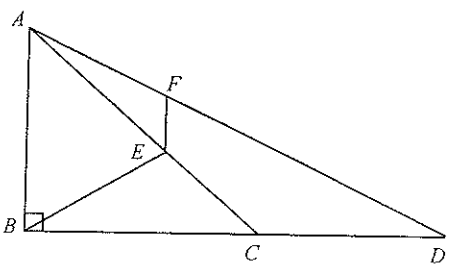
1. The digits 1, 2, 3, and 4 are to be placed into the boxes at the right in such a way that all four digits are in each row and all four are in each 2 by 2 box outlined with darker lines. In how many distinct ways can the numbers be placed in the boxes?



2. For how many integral values of n between 1 and 2009 inclusive is the following an integer?

$$\sqrt{n-\sqrt{n-\sqrt{n-\sqrt{n-\sqrt{\cdots}}}}}$$

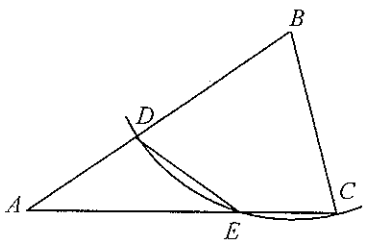
3. Starting at the base of a vertical light post $\overline{AB}$, Efe walks up inclined plank $\overline{BE}$, stopping at E. If $AB = 18$, $m\angle EBC = 30°$, $BE = 12$, and Efe's height, i.e., segment $\overline{EF}$, is 6 units in length, determine the number of units in the length of Efe's shadow on the floor, namely the length of $\overline{CD}$.



4. In circle O, $m\widehat{BA} = m\widehat{AC} = 60°$ and P lies on circle O not on minor $\widehat{BC}$. Find the value of the ratio $\dfrac{PC + PB}{PA}$.

5. For $x, y, z > 0$, if $x + \dfrac{1}{y} = a$, $y + \dfrac{1}{z} = b$, and $z + \dfrac{1}{x} = c$, find the minimum value of the product abc.

6. In an acute triangle ABC, $\overline{BC}$ is the shortest side and is taken as the radius of a circle with center B; this circle intersects $\overline{AB}$ at D and $\overline{AC}$ at E. If $AD = DE$, find all possible degree measures for angle A.



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Answer Sheet

Round 1

1. 4
2. 10
3. 28

Round 2

1. 3
2. 4
3. $(11,14), (12,13)$

Round 3

1. 3
2. $9-3\sqrt{3}$
3. $\left(\sqrt{3}-1\right):1$

Round 4

1. 5
2. 34
3. 574

Round 5

1. 12
2. $k > -25$
3. 6

Round 6

1. $\frac{\sqrt{11}}{6}$
2. $\frac{1+\sqrt{3}}{2}$
3. 16

Team

1. 96
2. 44
3. $9\sqrt{3}$
4. $\sqrt{3}:1$
5. 8
6. $30 < \theta < 45$