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April 2, 1013

Possible Solutions:

Round I: Arithmetic and Number Theory

1) The powers of 2 end in 2, 4, 8, and then 6. So we need to find the remainder from the division problem 2013 divided by 4.

$\frac{2013}{4} = 503 \text{ R1} \rightarrow 2^{2013}$ end in 2. Now 4(2) = **8**.

2)

$$1+2+3+\ldots+n = 25n \rightarrow \frac{n(n+1)}{2} = 25n$$

$$n^2 + n = 50n \rightarrow n^2 - 49n = 0 \rightarrow n = 49$$

3)

| + | 1 | 3 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 4 | 5 | 6 | **7** |
| 2 | 3 | 5 | 5 | 6 | **7** | **8** |
| 5 | 6 | **8** | **8** | **9** | **10** | **11** |
| 4 | 5 | **7** | **7** | **8** | **9** | **10** |
| 5 | 6 | **8** | **8** | **9** | **10** | **11** |
| 6 | **7** | **9** | **9** | **10** | **11** | **12** |

This give 24 possible wins out of 36. So the probability is 24/36 or **2/3.**

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Round II. Algebra I

1) With the absolute value you have two possibilities:
either 2x –3 = 9 –2x or –2x + 3 = 9 – 2x. The second choice give 3 = 9, which is not true so this portion gives no solution. Now simplify the first choice: giving 4x = 12 or **x = 3**.

2) Let n coins have the value v $\rightarrow \frac{v}{n} = 19 \rightarrow v = 19n$. If we add one quarter our equation become $\frac{v+25}{n+1} = 20$. Now substitute for v $\rightarrow \frac{19n+25}{n+1} = 20 \rightarrow 19n+25 = 20n+20 \rightarrow n = 5, v = 95$.
To get 95¢ you need 3 quarters and 2 dimes. So, you have **2** dimes.

3) Start with 5 = 3 + 2. Now the problem becomes

$$\sqrt{3+2\sqrt{6}+2} \rightarrow \sqrt{(\sqrt{3}+\sqrt{2})^2} \rightarrow \sqrt{3}+\sqrt{2}$$

Now, $x = 3, y = 2 \rightarrow x^2 + y^2 = 9 + 4 = 13$

CSAML
April 2, 1013

## Round III: Geometry

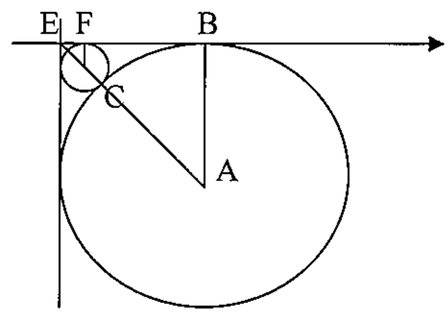
1) Notice that $\Delta ABC$ is a right triangle and now side AC will be 13. Let AE = 3x and AC = 5x, now 5x = 13, or x = 13/5. Also, by similar triangles: $\frac{AD}{DB}=\frac{3}{2};\frac{DE}{BC}=\frac{36/5}{12}$. To find the area of DECB, a trapezoid use

$$A=\frac{1}{2}(b_1+b_2)h \rightarrow \frac{1}{2}\left(\frac{36}{5}+\frac{36}{3}\right)\cdot 2$$

$$A=36\cdot\frac{8}{15}=\frac{96}{5}$$

2) Since the inscribed angle is 30°, the arc is 60°. Now the arc length is $\overset{\frown}{BC}=\frac{60}{360}(2\pi(6))=2\pi$

3) The large circle has a radius of 1 and its center is $\sqrt{2}$ units from the origin.




$$AE=1+r+r\sqrt{2}=\sqrt{2}$$

$$r(1+\sqrt{2})=\sqrt{2}-1\rightarrow r=3-2\sqrt{2}$$

$$A=\pi r^2=\pi\left(17-12\sqrt{2}\right)$$

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## Round IV: Algebra II

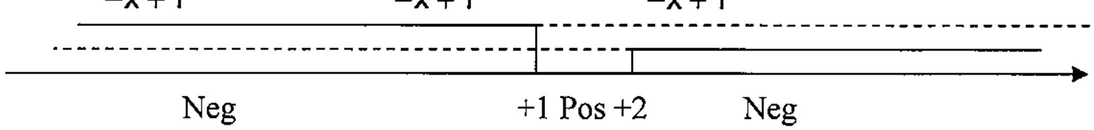
1)

$$\log_3\left(x^2+5x-6\right)-\log_3(1-x)=2\rightarrow \log_3\frac{(x-1)(x+6)}{(1-x)}=2$$

$$\log_3(-x-6)=2\rightarrow -x-6=3^2\rightarrow x=-15$$

2)

$$\frac{2x-6}{-x+1}-2\le 0\rightarrow\frac{2x-6+2x-2}{-x+1}\le 0\rightarrow\frac{4(x-2)}{-x+1}\le 0$$




So, the result is neg or zero for $x<1$ or $x\ge 2$ or $(-\infty,1)\cup[2,\infty)$

3) If one root is $\left(\frac{4}{5}-\frac{3}{5}i\right)$, then there is another root that is $\left(\frac{4}{5}+\frac{3}{5}i\right)$. These two roots are the quadratic $5x^2-8x+5$. Now divide the given equation by this quadratic and the quotient is $2x^2-5x+2$ whose zeroa are ½ and 2. Thus the remaining (three) roots are $\left(\frac{4}{5}+\frac{3}{5}i\right)$, 2 and ½.

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Round V: Analytic Geometry

1) The radius of the circle is perpendicular to the given line. The radius is on a line with slope of +2. Substitute the center of the circle to find the equation of the line containing the radius and you get

$y=2x+b$ $\qquad$ $y = 2x+b$

$1=2(8)+b \rightarrow b=-15$ $\qquad$ $1=$ Solve the two lines simultaneously:

$y=2x-15$ $\qquad$ $y=$ $\begin{cases} y=-\frac{1}{2}x \\ y=2x-15 \end{cases} \rightarrow 2x-15=-\frac{1}{2}x \rightarrow 4x-30=-x \rightarrow 5x=30 \rightarrow x=6, y=-3$

2) The given parabola can be written in the form: $y^2-4y+4=-4x+4 \rightarrow (y-2)^2=-4(x-1)$. This gives a parabola with vertex at (1, 2) and focus at (0, 2). Now the circle will have a radius of 1 and center at (0, 2). $x^2+(y-2)^2=1$.

3) Substitute the two points into the equation

$$\begin{cases} \frac{8}{a^2}+\frac{3}{b^2}=1 \\ \frac{10}{a^2}+\frac{9/4}{b^2}=1 \end{cases} \rightarrow \begin{cases} \frac{40}{a^2}+\frac{15}{b^2}=5 \\ \frac{40}{a^2}+\frac{9}{b^2}=4 \end{cases} \rightarrow \frac{6}{b^2}=1 \rightarrow b^2=6 \rightarrow a^2=16 \rightarrow 2a=8.$$

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Round VI: Trigonometry, Complex numbers

1) Note that

$i^{2010}=i^2=-1$

$i^{2011}=i^3=-i$

$i^{2012}=i^4=1$

$i^{2013}=i$

So, $2010\cdot i^{2010}+2011\cdot i^{2011}+2012\cdot i^{2012}+2013\cdot i^{2013}$
$=-2010-2011\,i+2012+2013\,i=2+2\,i$

CSAML
April 2, 1013

2)

$$(\sin\theta + \cos\theta)^2 = \frac{1}{2} \rightarrow \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = \frac{1}{2}$$

$$1 + 2\sin\theta\cos\theta = \frac{1}{2} \rightarrow 2\sin\theta\cos\theta = -\frac{1}{2} \rightarrow \sin 2\theta = -\frac{1}{2}$$

$$2\theta = 210^\circ, 330^\circ, 570^\circ, 690^\circ$$

$$\theta = 105^\circ, 165^\circ, 285^\circ, 345^\circ$$

3) If you use DeMoivre's Theorem the problem can be stated as $\sqrt[3]{\square} = 2\text{cis}50^\circ$.
To find $Z$:

$$\left(Z^{1/3}\right)^3 = 2^3 \cos 3(50^\circ)$$

$$Z = 8\cos 150^\circ = 8(\cos 150^\circ + i\ \sin 150^\circ)$$

$$Z = 8\left(-\frac{\sqrt{3}}{2} + i\ \cdot \frac{1}{2}\right)$$

$$a + b = -4\sqrt{3} + 4$$

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TEAM ROUND

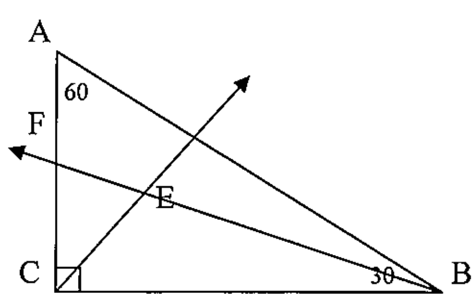
1) To begin, there are 6(5)(4)(3) = 360 ways for the boys to occupy the the corner seats. Now the remaining seats have 4(3)(2)(1) = 24 possible arrangements of the four people.Using the multiplication principle you get **8640 way to be seated.**.

2) There are 2 cases. Case I: x + y + z = 77 and z – y = 4, z – x = 9, and y – x = 9. Substitute for x and y and you have z – 9 + z – 4 + z = 77, z = 30, y = 26 and x = 21.

Case II: x + y + z = 77 and z – y = 5, z – x = 9 and y – x = 4. Substitute for x and y and get z – 5 + z – 9 + z = 77 or 3z = 91. In this case z is not an integer, so there is no solution in this case.

The only answer is **x = 21, y = 26, z = 30.**

3)




Let BE = x and EF = 1. The sides of the triangle are AB = 2a, AC = a and BC = $a\sqrt{3}$. $\overline{BF}$ is the bisector of $\angle$B so, $AF / FC = 2a / \sqrt{3}a = 2 / \sqrt{3}$.Of course, FC + AF = a, so FC $+\left(2/\sqrt{3}\right)$FC = a, or $FC = \frac{a\sqrt{3}}{\sqrt{3}+2}$. Now using the bisector of $\angle$C we have

$$\frac{FC}{1} = \frac{BC}{x} \rightarrow \frac{a\sqrt{3}}{\sqrt{3}+2} = \frac{a\sqrt{3}}{x} \rightarrow x = \sqrt{3} + 2$$

CSAML

April 2, 1013

4) Let $f(x) = ax^2 + bx + c$. Expand Now substitute into the original

$5f(x) - 3f(x-2) =$

:

$5ax^2 + 5bx + 5c - 3ax^2 + 12ax - 12a - 3bx + 6b - 3c = 4x^2 + 4$

$x^2(5a - 3a) + x(5b + 12a - 3b) + (-12a + 6b + 3c) = 4x^2 + 4$

Group the terms to find a, b, and c.

$5a - 3a = 4 \to a = 2$

$5b + 12a - 3b = 0 \to b = -12$

$5c - 12a + 6b - 3c = 4 \to c = 32$

$f(x) = 2x^2 - 12x + 50$

$f(3) = 18 - 36 + 50 = 32$

5) After you draw the picture you can set up a coordinate system with the origin at point D. Now the circle center at M (2, 0) will have the equation $(x-2)^2 + y^2 = 4$ and the circle with center at A (0, 4) will be $x^2 + (y-4)^2 = 16$. Solving for the point P, $\begin{cases} x^2 + (y-4)^2 = 16 \\ (x-2)^2 + y^2 = 4 \end{cases} \to x = 2y$. Now by substitution x = 16/5 and y = 8/5. The distance P is from $\overline{AD}$ is **16/5.**

6) Since $0 < x < 90$, sin x > 0 and cos x > 0. Now $\sin^2 x > 1 - \sin 2x \to \sin^2 x + 2\sin x\cos x - 1 > 0$. Divide by $\cos^2 x$ to get $\tan^2 x + 2\tan x - \sec^2 x > 0 \to 2\tan x - 1 > 0 \to \tan x > 1/2 \to k = 1/2$.