CSAML
April 3, 2012

Possible solutions:
Round I: Arithmetic and Number Theory

1. The pattern has 5 directional arrows, so 5n + R = 2012.
   5(402) + 2 = 2012, so the 2^{nd} leg is the 2012^{th}.

2. S:D:P=11:1:90 or (a + b):(a - b): ab = 11:1:90

$$\begin{cases} a+b=11x \\ a-b=x \\ ab=90x \end{cases} \rightarrow a+b=11a-11b \rightarrow a=\frac{6b}{5}, x=\frac{b}{5}$$

$$\frac{6b^2}{5}=(90)\left(\frac{b}{5}\right) \rightarrow b=15, a=18 \rightarrow ab=270$$

3. The last 2 digits of 4^n has a 10 cycle repeat:

<u>4</u>, 16, 64, 256, 1024, 4096, 16384, 65536, 262144, 104856, 4194<u>304</u>, ..So, $\frac{6036}{10}=603+\frac{6}{10}$.
Now the last 2 digits of our problem are 96, and the sum of these two digits is <u>15</u>.

Round II. Algebra I

1.

$$|||x-1|-1|-1|=1 \rightarrow ||x-1|-1|-1=1 \rightarrow ||x-1|-1|=2 \text{ or } ||x-1|-1|=0$$

$$||x-1|-1|=2 \rightarrow |x-1|-1=\pm 2 \rightarrow |x-1|=3 \text{ or } |x-1|=-1$$

$$|x-1|=3 \rightarrow x-1=\pm 3 \rightarrow x=4 \text{ or } x=-2$$

$$||x-1|-1|=0 \rightarrow |x-1|-1=0 \rightarrow |x-1|=1 \rightarrow x-1=\pm 1 \rightarrow x=2 \text{ or } x=0$$

2.

$$\begin{cases} 3A+B+0=61 \\ 0+3B-C=1 \\ A+0+3C=26 \end{cases} \rightarrow \begin{cases} 0+9B-3C=3 \\ A+0+3C=26 \end{cases} \rightarrow A+9B=29$$

$$\begin{cases} 3A+B=61 \\ A+9B=29 \end{cases} \rightarrow \begin{cases} 3A+B=61 \\ 3A+27B=87 \end{cases} \rightarrow 26B=26 \rightarrow =1$$

$$3A+B=61 \rightarrow 3A=60 \rightarrow A=20$$

$$A+0+3C=26 \rightarrow 3C=6 \rightarrow C=2$$

$$(A,B,C)=(20,1,2) \rightarrow (x,y,z)=\left(\frac{1}{20},1,\frac{1}{2}\right)$$

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3. Rewrite the problem using fractional exponents.

$(x+8)^{2/3}-2(x+8)^{1/3}-24=0$

Let $A=(x+8)^{1/3} \rightarrow A^2-2A-24=0$

$(A+4)(A-6)=0$

$\sqrt[3]{x+8}=-4 \rightarrow x+8=-64 \rightarrow x=-72$

$\sqrt[3]{x+8}=6 \rightarrow x+8=216 \rightarrow x=208$

Round III: Geometry

1. I chose x = the hypotenuse. Now, one leg is (x – 2) and the other leg is (x-1)/5.
By the Pythagorean Theorem:

$\frac{(x-1)^2}{25}+(x-2)^2=x^2 \rightarrow \frac{x^2-2x+1}{25}+x^2-4x+4=x^2$

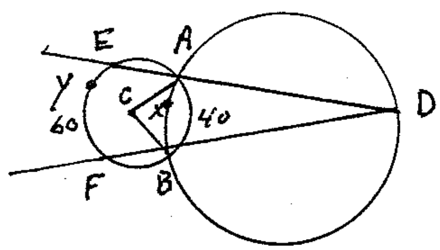
$x^2-2x+1=100x-100 \rightarrow x^2-102x+101=0$

$(x-1)(x-101)=0 \rightarrow x=101$

2.

$m\angle D=\frac{60-40}{2}=10$

$m\angle D=\frac{1}{2}m\widehat{AXB} \rightarrow m\widehat{AXB}=20$



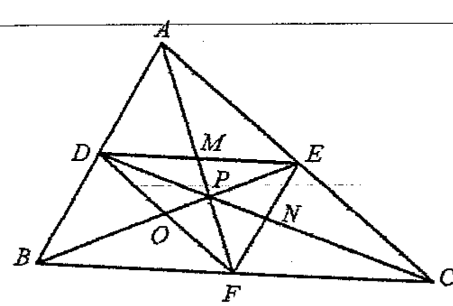
3.

Since M is the midpoint of $\overline{DF}$, then $MF=\frac{1}{2}AF$ and since $\overline{FM}$ is a median of $\triangle DEF$,

then $MP=\frac{1}{3}MF \rightarrow MP=\frac{1}{6}AF$.

Likewise, $NP=\frac{1}{6}CD$ and $OP=\frac{1}{6}BE$. So $MP+NP+OP=\frac{1}{6}(108)=18$

Round IV: Algebra II



1. $0<\ln x\leq 1,\ \ln x=\log_e x=1 \rightarrow x=e^1$

$0<x\leq e$

2. $\begin{cases} P(-1) = a + b + 2010 = 5 \\ P(1) = a - b + 2010 = 15 \end{cases} \rightarrow 2a + 4020 = 20 \rightarrow a = -2000, b = -5 \rightarrow (-2000, -5)$

3.

$$m = \frac{a+2}{5-a} \geq 2 \rightarrow \frac{a+2}{5-a} - 2 \geq 0$$

$$\frac{a+2}{5-a} - \frac{2(5-a)}{(5-a)} \geq 0 \rightarrow \frac{3a-8}{5-a} \geq 0$$

$$\frac{8}{3} \leq a < 5 \text{ or } \left[\frac{8}{3}, 5\right)$$

Round V: Analytic Geometry

1. Substitute each given point into the equation and find b.
Given:

$$\frac{(x-5)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$$

$$\text{with } (0,0)\ \frac{25}{a^2} + \frac{9}{b^2} = 1$$

$$\text{with } (7,3)\ \frac{4}{a^2} + 0 = 1 \rightarrow a^2 = 4$$

$$\text{Now, } \frac{25}{4} + \frac{9}{b^2} = 1 \rightarrow b^2 = \frac{36}{21} \rightarrow b = \frac{2\sqrt{21}}{7}$$

2. First find the center of the ellipse and the value of a.

$$3x^2 - 18x + \_\_\_\_ + 2y^2 + 16y + \_\_\_\_ = -58$$

$$3(x^2 - 6x + 9) + 2(y^2 + 8y + 16) = -58 + 3(9) + 2(16)$$

$$3(x-3)^2 + 2(y+4)^2 = 1 \rightarrow \frac{(x-3)^2}{1/3} + \frac{(y+4)^2}{1/2} = 1$$

Notice that $\frac{1}{2}$ is greater than $\frac{1}{3}$, so the major axis is parallel to the y-axis and the ends of the major axis are at $\left(3, -4 \pm \frac{\sqrt{2}}{2}\right)$.

3. The distance from the point (x, y) to (-4, -3) is equal to the distance from the point (x, y) to (-3, 4). So,

$$\sqrt{(x+3)^2+(y-4)^2}=\sqrt{(x+4)^2+(y+3)^2}$$

$$x^2+6x+9+y^2-8y+16=x^2+8x+16+y^2+6y+9$$

$$-2x=14y \rightarrow x=-7y$$

$$x^2+y^2=25 \rightarrow 49y^2+y^2=25 \rightarrow y=\pm\frac{\sqrt{2}}{2} \rightarrow x=\mp\frac{7\sqrt{2}}{2}$$

$$\left(\frac{7\sqrt{2}}{2},\frac{-\sqrt{2}}{2}\right),\left(\frac{-7\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$

Round VI – Trigonometry, Complex numbers.

1. $\frac{\tan x}{2\sin x}=-1 \rightarrow \frac{1}{\cos x}=-2 \rightarrow \cos x=-\frac{1}{2} \rightarrow\rightarrow x=\frac{2\pi}{3},\frac{4\pi}{3}$

2. When you joint the vertices F and H, you form an isosceles triangle. Notice that a dodecagon has an interior angle of $150°$. Now you can find the square of the distance FH since it is the 3rd side of the triangle. Using the law of cosines:

$$x^2=9^2+9^2-2(9)(9)\cos 150°$$

$$x^2=162-2(81)\left(-\frac{\sqrt{3}}{2}\right)=162+82\sqrt{3}$$

3. The cos 3x is at a minimum value when $3x=\pi+2\pi n$ for n = 0, 1, 2, ...

$$x=\frac{\pi}{3}+\frac{2\pi n}{3}$$

$$0\le\frac{\pi}{3}+\frac{2\pi n}{3}<2\pi$$

$$\frac{\pi}{3},\pi,\frac{5\pi}{3}$$

Team Round

1. There are 100 9's in the units position, 99 9's in the tens position, 98 9's in the hundreds place and 97 9's for the thousands place. The sums is 900 + 9(990) + 9(9800) + 9(97000). Add these values up to the 4 necessary digits. 1010

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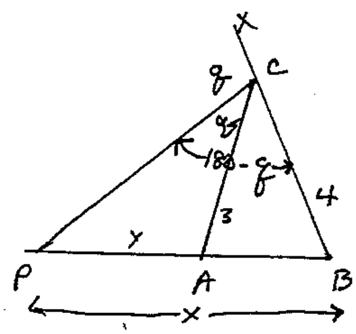
2. Multiply by the LCM = b(b + x)(b – x) and get:

$(b+x)(b+x)(b-x)+2x(b)(b-x)+x^2(b+x)=2b(b+x)(b-x);\quad b\neq\pm b, b\neq 0$

$b^3-bx^2+b^2x-x^3+2b^2x-2bx^2+bx^2+x^3=2b^3-2bx^2$

$3b^2x=b^3\rightarrow x=\dfrac{b}{3}$



3. Using trigonometry:

$\dfrac{4}{\sin\theta}=\dfrac{x}{\sin(180-q)}=\dfrac{x}{\sin q}$ and $\dfrac{3}{\sin\theta}=\dfrac{y}{\sin q}$

$\dfrac{3}{y\sin\theta}=\dfrac{4}{x\sin\theta}\rightarrow\dfrac{x}{y}=\dfrac{4}{3}\rightarrow\dfrac{y}{x-y}=\dfrac{3}{1}$

Using Geometry:

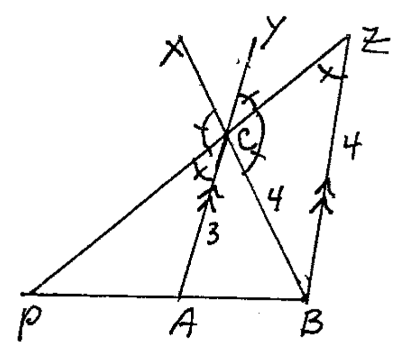
Draw

$BZ\parallel AC, \angle XCP\cong\angle PCA\cong\angle Z$

$m\angle YCZ=m\angle ZCB=q$

$\Delta PAC\approx\Delta PBZ\rightarrow\dfrac{3}{AP}=\dfrac{4}{PB}\rightarrow\dfrac{PA}{PB}=\dfrac{3}{4}$

$\dfrac{PA}{AB}=\dfrac{3}{1}$



4.

$27x^3-y^3+3y^2-3y+1$

$(3x)^3-(y-1)^3$

$(3y-y+1)(9x^2+3xy-3x+y^2-2y+1)$

5.

$$\begin{cases}(x-3)^2+(y+2)^2=17^2\\ y+19=\dfrac{1}{2}(x-3)^2\end{cases}\rightarrow(x-3)^2=17^2-(y+2)^2$$

$2(y+19)=17^2-(y+2)^2$

$2(y+19)+(y+2)^2=17^2$

(CONT.)

$2y + 38 + y^2 + 4y + 4 = 289$

$y^2 + 6y + \_\_\_\_ = 289 - 42 + \_\_\_\_$

$y^2 + 6y + 9 = 247 + 9 = 256 = 16^2$

$(y+3)^2 = 16^2 \rightarrow (y+3) = \pm 16 \rightarrow y = \rightarrow 13, -19$

$(x-3)^2 = 2(13) + 38 \rightarrow (x-3) = \pm 8 \rightarrow x = 11, -5$

$(11,13), (-5,13)$

$(x-3)^2 = 2(-19) + 38 \rightarrow x = 3 \rightarrow (3,-19)$

A triangle is formed by these three points. The base of the triangle (11, 13) and (-5, 13) has a length of 16 and the height is [13 – (-19)] or 32, so the area is 16(32)/2 = 256.

6. The polynomial has integral coefficients so the least degree must be 4 with roots $\left(\frac{1}{2} + \frac{i\sqrt{7}}{3}\right), \left(\frac{1}{2} - \frac{i\sqrt{7}}{3}\right), (3-2i), (3+2i)$. Notice that we have pairs of roots of the form a + b*i* and a – b*i*. If we multiply $(x - a - bi)(x - a + bi) \rightarrow x^2 - 2ax + (a^2 + b^2)$. We can use this form with our given roots to get

$a = \frac{1}{2}, b = \frac{\sqrt{7}}{3} \rightarrow x^2 - 2\left(\frac{1}{2}\right)x + \left(\frac{1}{4} + \frac{7}{9}\right) \rightarrow 36x^2 - 36x + 37 = 0$ and when

$a = 3, b = -2 \rightarrow x^2 - 2(3)x + (9+4) \rightarrow x^2 - 6x + 13$. Now multiply these to find the coefficients.

$(x^2 - 6x + 13)(36x^2 - 36x + 37) \rightarrow$

$(36x^4 - 36x^3 + 37x^2) + (-216x^3 + 216x^2 - 222x) + (468x^2 - 468x + 481)$

$36 - 36 + 37 - 216 + 216 - 222 + 468 - 468 + 481 = 296$