FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

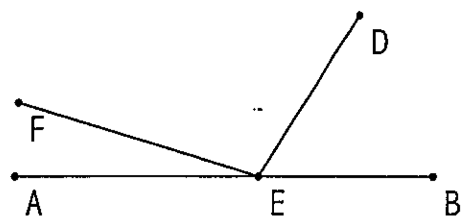
Match 6 Round 1
Geometry: Lines and Angles

1.) ____81 degrees____

2.) ____150 degrees________

3.) ____ $\frac{140+5x}{2}$ ________

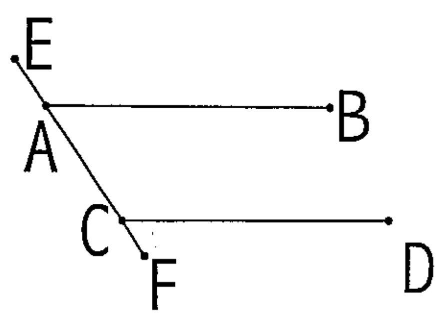
1) Point E is on line AB. $m\angle AEF$ is 9 degrees less than twice the measure of the complement of $\angle DEB$ and is one-fourth the measure of $\angle AED$. Find the measure of $\angle DEF$ in degrees.



Let x = the measure of angle DEB. The measure of angle AEF is 2(90-x)-9 = 171-2x. Since AED is supplementary to angle DEB, the measure of angle AEF is also (1/4)(180-x). Solve 171-2x = $\frac{180-x}{4}$, so 684-8x=180-x, so 7x=504, and x=72 degrees. Then the measure of angle AEF is 27 degrees, so the measure of angle DEF is 180-99 = 81 degrees

2) Lines AB and CD are parallel and are cut by transversal AC as shown.

If $m\angle BAE$ = (x^2 +50) degrees and $m\angle DCF$ =(2x+10) degrees, find all possible values

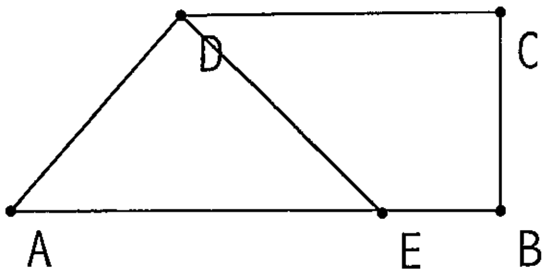


of $m\angle BAE$ in degrees

The two angles must be supplementary, so (x^2 + 50) + (2x+10) = 180.

X^2 + 2x -120 = 0, so (x+12)(x-10) = 0, so x=-12 or x=10. However, x can't be -12, since that would give angle DCF a negative measure, so x must be 10, so x^2+50 = 150 degrees.

3) In trapezoid ABCD, AB is perpendicular to BC. A ray from D bisects angle ADC and meets side AB at E. If the measure of $\angle DAB$ is (5x-40) degrees, find the measure of $\angle BED$ in terms of x. Express your answer as a single fraction involving x.



Let x=Since CB is perpendicular to AB, angle DAE is supplementary to angle ADC. Angle ADC = 180-(5x-40) = 220-5x, so angle ADE is (220-5x)/2, and so is angle CDE since ray DE is a bisector. Angle CDE and BED must also be supplementary, so the measure of angle DEB is 180-(220-5x)/2 = (360-220+5x)/2 = (140+5x)/2

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 6 Round 2
Algebra: Literal Equations

1.) ___a=b(y+x) alternatively a=by+bx

2.) _$x=-2t-\frac{7}{2}$, alternatively $x=\frac{-4t-7}{2}$

3.) _____$p^2 or \frac{p^2}{2}$ ________________

1) If y≠x, solve the following for a and simplify as much as possible:

$ax+by^2=bx^2+ay$ $\frac{3p}{}$

$by^2-bx^2=ay-ax$

$b(y^2-x^2)=a(y-x)$

$\frac{y^2-x^2}{y-x}=\frac{a}{b}$

$a=b(y+x)$

alternatively a=by+bx

2) If $y=\frac{2t-3}{4}$ and $5t=6y-x+1$, solve for x in terms of t.

$5t=6(\frac{2t-3}{4})-x+1$

$5t=3t-\frac{9}{2}-x+1$

$2t=-x-\frac{7}{2}$

$2t+\frac{7}{2}=-x$

$x=-2t-\frac{7}{2}$

alternatively x= $\frac{-4t-7}{2}$

3) Solve for all possible expressions of x in terms of p and simplify each as much as possible: $3x^2+(2p^2)^2-2p^2x=3p^4+p^2x+x^2$

$2x^2-3p^2x+p^4=0$, so $x=\frac{3p^2\pm\sqrt{9p^4-4*2*p^4}}{4}=\frac{3p^2\pm p^2}{4}$

$\frac{3p^2\pm p^2}{4}=\frac{4p^2}{4} or \frac{2p^2}{4}=p^2$ or $\frac{p^2}{2}$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 6 Round 3
Geometry:
Solids and
Volumes

1.) $\frac{256\pi}{3}$ alternatively $(85\frac{1}{3})\pi$

2.) $\frac{1000\sqrt{3}}{9}$

3.) $\frac{10\sqrt[3]{9}}{3}$

1) A spherical tennis ball of radius 4 cm is placed on an open cylindrical can of height 8 cm and base radius 4 cm so that half of the tennis ball remains outside of the can. Find the volume of the can that is not taken up by the tennis ball. Express your answer as a single fraction.

The volume of the cylinder is $\pi(4^2)8 = 128\pi$. The volume of half the sphere is $\frac{2}{3}\pi * 4^3 = \frac{128\pi}{3}$ The difference is $\frac{384\pi - 128\pi}{3} = \frac{256\pi}{3}$

2) A sphere of radius 5 cm is circumscribed about a cube. Find the volume of the cube.

Two diametrically opposite points of the sphere join through the longest diagonal of the cube. If the cube has side s, a diagonal across one face has side $\sqrt{s^2+s^2} = s\sqrt{2}$, and then the longest diagonal has length $\sqrt{(s\sqrt{2})^2+s^2} = s\sqrt{3}$. The radius of the sphere is then $\frac{s\sqrt{3}}{2}$. If $\frac{s\sqrt{3}}{2} = 5$, then s $= \frac{10\sqrt{3}}{3}$. The volume of the cube is

$$s^3 = (\frac{10\sqrt{3}}{3})^3 = \left(\frac{1000*3\sqrt{3}}{27}\right) = \frac{1000\sqrt{3}}{9}$$

3) A cone is formed by rotating the line segment from (0,0) to (5,10) around the y-axis. The line y=k splits the cone so that the part of the volume of the original cone above y=k is twice the part of the volume of the original cone below y=k. Find the value of k.

The base radius of the original cone is 5 and the height is 10, so the volume of the original cone is $\frac{1}{3}\pi(5^2)(10) = \frac{250\pi}{3}$. When the plane at y=k cuts the cone, a new cone is created by rotating the point $\left(\frac{k}{2}, k\right)$ around the y-axis, so its volume will be

$\frac{1}{3}\pi(\frac{k}{2})^2(k) = \frac{k^3\pi}{12}$. We need $\frac{k^3\pi}{12} = \frac{1}{3} * \frac{250\pi}{3}$, so $9k^3=250*12=3000$, so

$$k = \sqrt[3]{\frac{3000}{9}} = \sqrt[3]{\frac{9000}{27}} = \frac{10\sqrt[3]{9}}{3}$$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 6 Round 4
Radical Expressions and Equations

1.) ________ $\frac{26\sqrt{3}}{3}$ ____

2.) ________ $\sqrt[6]{5000}$ ________

3.) ________ 0, 8 ________

1) Express in simplest radical form:

This is $5*4\sqrt{3}+\frac{30}{5\sqrt{3}}-\frac{48*5}{6\sqrt{3}} = 20\sqrt{3}+\frac{6}{\sqrt{3}}-\frac{40}{\sqrt{3}}=$

$$20\sqrt{3}+\frac{6}{\sqrt{3}}-\frac{40}{\sqrt{3}}=20\sqrt{3}-\frac{34\sqrt{3}}{3}=\frac{60\sqrt{3}-34\sqrt{3}}{3}=\frac{26\sqrt{3}}{3}$$

2) Express $\sqrt[3]{25}$ times $\sqrt{2}$ as a single radical in the form $\sqrt[n]{a}$ for the smallest possible whole number values of n and a where $\sqrt[n]{a}$ is in simplest radical form.

Write $\sqrt[3]{25}=5^{\frac{2}{3}}=5^{\frac{4}{6}}$ and $\sqrt{2}=2^{\frac{3}{6}}=8^{\frac{1}{6}}$. Then multiply $5^{\frac{4}{6}}*8^{\frac{1}{6}}=(625*8)^{\frac{1}{6}}=\sqrt[6]{5000}$

3) Solve the equation $\sqrt{10x+1}-\sqrt{2x}=\sqrt{3x+1}$ for all possible real values of x.

$\sqrt{10x+1}=\sqrt{2x}+\sqrt{3x+1}$

$(\sqrt{10x+1})^2=(\sqrt{2x}+\sqrt{3x+1})^2$

$10x+1=2x+2\sqrt{2x}\sqrt{3x+1}+3x+1$

$5x=2\sqrt{2x}\sqrt{3x+1}$

$25x^2=4*2x*(3x+1)$

$25x^2=24x^2+8x$

$x^2-8x=0,$

so x=8 or x=0, and neither is extraneous.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 6 Round 5 Polynomials and Advanced Factoring

1.) __ $x^3-6x^2+13x-20$____

2.) _$(2a-3b)(4a^2+6ab+9b^2)(2a+3b)(4a^2-6ab+9b^2)$

3.) __ $(x^2+1)(x+8)(x-6)$______

1) A cubic polynomial with integer coefficients has 4 and 1+2i as two of its zeros, where $i = \sqrt{-1}$. Express the polynomial in the form $ax^3 + bx^2 + cx + d$ for integers a,b,c, and d where a,b, c, and d are relatively prime and a>0
Since the polynomial has integer coefficients, if 1+2i is a zero, so is 1-2i. The sum of 1+2i and 1-2i is 2, and the product of 1+2i and 1-2i is 5, so they are the solutions to the quadratic equation $x^2-2x+5=0$. The polynomial is $(x-4)(x^2-2x+5) = x^3-6x^2+13x-20$

2) Give the complete factoring of $64a^6 - 729b^6$ as four factors with integer coefficients. It is possible but tricky if you approach this as the sum of two cubes. Easier to approach it as the sum of two squares. $(8a^3)^2 - (27b^3)^2 = (8a^3-27b^3)(8a^3+27b^3) = (2a-3b)(4a^2+6ab+9b^2)(2a+3b)(4a^2-6ab+9b^2)$

3) Give the complete factoring of the polynomial $x^4 + 2x^3 - 47x^2 + 2x - 48$ given that all factors have integer coefficients.
Rewrite this as $(x^4+2x^3-48x^2)+(x^2+2x-48) = x^2(x^2+2x-48) + 1(x^2+2x-48) = (x^2+1)(x^2+2x-48) = (x^2+1)(x+8)(x-6)$

FAIRFIELD COUNTY
MATH LEAGUE (FCML) 2012-2013

Match 6 Round 6
Counting and Probability

1.) ________2________

2.) ________455________

3.) ________3744_____

1) Debate Club, Ecology Club, and French Club each consist of 17 students. There are 5 students who belong to both Debate Club and Ecology Club, 6 students who belong to both Debate Club and French Club, and 7 students who belong to both Ecology Club and French Club. If 21 students belong to exactly one club, how many students belong to all three clubs?
Draw out a Venn diagram with the three circles overlapping in the center. If x is the number of students belonging to all 3 clubs, then 5-x belong to D and E alone, 6-x belong to D and F alone, and 7-x belong to E and F alone. Since 21 students belong to exactly one club, and each club must have the same number of students (17), the 21 must be split up so that 8 students belong to D alone, 7 students to E alone, and 6 students to F alone. Any of the circles now has a total of 19-x. If 19-x =17, x=2.

2) Fourteen different ping-pong balls numbered 1 through 14 are placed in a bin and four balls are drawn randomly. The balls are not replaced after they are drawn. How many different combinations have at least two balls with two digits? Solution: 5 of the balls have two digits on them and 9 of them have only 1. The number of ways you can draw exactly 2 balls with exactly two digits is $({}_9C_2)({}_5C_2)$. The number of ways you can draw 3 balls with 2 digits is $({}_9C_1)({}_5C_3)$. The number of ways you can draw 4 balls with 2 digits is $({}_9C_0)({}_5C_4)$.
The solution is 36*10 + 9*10 + 1*5 = 360+90+5 = 455.

3) Three cards are drawn without replacement from a standard 52-card deck of 13 different denominations and 4 different suits. Define a "half-house" as drawing 2 cards of one denomination and 1 card of a different denomination. The order in which the cards are drawn does not matter. How many different half-houses are possible?
Getting 2 jacks and 1 queen is a different half-house from getting 2 queens and 1 jack, so although the order in which the cards is drawn does not matter, there are 13*12 ways to choose the first denomination that has two cards and then the second denomination that has one card. Take the denomination that has 2 cards. We need to choose 2 of those out of 4 that are in the deck, so multiply by $({}_4C_2)$. Then for the denomination with 1 card, we need to pick of one of the 4 cards with that denomination, so multiply by $({}_4C_1)$. 13*12*6*4 = 156*24 = 3744

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 6 Team Round

1.) ______72___________ 4.) _______ $10+6\sqrt{3}$_____

2.) ___x=a-11________5.) _$(4a^2 + 5b^2 + 3ab)(4a^2 + 5b^2 - 3ab)$_

3.) __$\frac{1000\pi}{3}$__________ 6.) __________$\frac{3}{8}$__________

1) The measure of angle A is 3 times the square root of the measure of the complement of angle B. The measure of angle B is 3 times the measure of angle A. Find the sum of the measures of angle A and angle B.

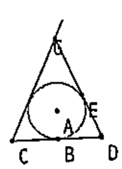
Let a = measure of angle A and b=measure of angle B.
$a=3\sqrt{90-b}$ and b=3a. Substitute to get $a = 3\sqrt{90-3a}$. Square both sides to get $a^2 = 9(90-3a)$, so $a^2+27a-810=0$. This factors to $(a-18)(a+45)=0$, so a=18. 3a=54, so a+b=72.

2) Given $a\neq-9$, $x\neq2$, and $a>13$, express x in terms of a and simplify as much as possible:

$$ax^2+9x^2-4a=a^2x-81x-2a^2+198$$

Rewrite this as $ax^2+9x^2-4a-36=a^2x-81x-2a^2+162$. Factor each side by grouping to get $(a+9)(x+2)(x-2) = (a+9)(a-9)(x-2)$. Cancel common factors of a+9 and x-2 to get x+2 = a-9, so x=a-11. One of the provers wanted to add the a>13 criterion if you use quadratic formula instead of factoring so that there is no ambiguity about which root of $\sqrt{(a-13)^2}$ to use.

3) A sphere of radius 5 cm is inscribed in a right circular cone of height 20 cm. Find the volume of the cone in cubic cm.



Look at a cross section of the figure.

AB=5, so AG=15, and AE=5, so by Pythagorean Theorem, GE=$\sqrt{15^2-5^2}=10\sqrt{2}$.

ΔAGE is similar to ΔDGB, so $\frac{AE}{DB}=\frac{GE}{GB}$, $\frac{5}{DB}=\frac{10\sqrt{2}}{20}$, so DB=$\frac{100}{10\sqrt{2}}=\frac{10}{\sqrt{2}}=5\sqrt{2}$.

The volume is $\frac{1}{3}\pi(5\sqrt{2})^2(20)=\frac{1000\pi}{3}$

4) One solution to the equation $\sqrt[3]{x}=\sqrt{\frac{x+2}{3}}$ is x=1. Find the other real solution.

Rewrite as $\sqrt{3}\sqrt[3]{x}=\sqrt{x+2}$. Raise both sides to the sixth power to get $27x^2= x^3+6x^2+12x+8$. Set one side=0 to get $x^3-21x^2+12x+8=0$. Since 1 is a solution, divide by x-1 to get $x^2-20x-8=0$. Solve by quadratic formula to get $x=\frac{20\pm\sqrt{432}}{2}=\frac{20\pm12\sqrt{3}}{2}=10\pm6\sqrt{3}.$ Since $6\sqrt{3}>10$, the negative solution is extraneous since the left side would lead to a negative answer while the right side is a positive answer, so the other solution is $10+6\sqrt{3}.$

5) Factor into two polynomials with integer coefficients: $16a^4+31a^2b^2+25b^4$
Rewrite this as $16a^4+40a^2b^2+25b^4-9a^2b^2=(4a^2+5b^2)^2-(3ab)^2=$
$(4a^2+5b^2+3ab)(4a^2+5b^2-3ab)$

6) A group of 4 students put their calculators into a pile, and then they each randomly choose a calculator from the pile. What is the probability that no student gets his or her own calculator?
There are 24 possible permutations. 6 of them have student 1 getting calculator 1, so eliminate those, and there are 18 left. Now 6 permutations have student 2 getting calculator 2, but we have already eliminated 1 2 3 4 and 1 2 4 3, so we only eliminate 4 more permutations, leaving 14. The remaining permutations are

2 1 3 4	4 1 3 2
2 1 4 3	4 3 1 2
3 4 2 1	4 3 2 1
3 4 1 2	2 4 3 1
3 1 2 4	2 4 1 3
3 1 4 2	2 3 1 4
4 1 2 3	2 3 4 1

Out of these, 2 1 3 4, 4 1 3 2, and 2 4 3 1 have student 3 getting calculator 3, so remove 3 more to leave 11. Out of the remaining permutations, 3 1 2 4 and 2 3 1 4 have student 4 getting calculator 4, so remove 2 more to leave 9. The answer is 9 out of 24, or $\frac{3}{8}$