FAIRFIELD COUNTY MATH LEAGUE 2017-2018

Match 4 Round 1 Arithmetic: Basic Statistics

1.) ________3.286________

2.) _______ 819____________

3.) ________20___________

1.) The scores of an AP test for a class are reported as follows: Six 5's, ten 4's, nine 3's, eight 2's, and 2 1's. Give the mean score rounded to three decimal places.

$$\frac{6*5+10*4+9*3+8*2+2*1}{35}=$$

$$\frac{30+40+27+16+2}{35}=\frac{115}{35}=\frac{23}{7}=3.2857...$$

$$rounds\_to\_3.286$$

2.) The upper quartile of a set of data is the median of the upper half of the data and the lower quartile is the median of the lower half of the data. The inter-quartile range is the upper quartile minus the lower quartile. Give the interquartile range of the set of the twelve smallest perfect cubes of natural numbers.

Numbers are 1,8,27,64,125,216,343,512,729,1000,1331,1728

Lower quartile is $\frac{27+64}{2}=45.5$. Upper quartile is $\frac{729+1000}{2}=864.5$. Difference is 864.5-45.5 = 819.

3.)_ The Math League bus costs $200 for one match. X students are planning to come to the match. If 4 of those students decide not to go, the mean cost per student increases by $2.50. Find X.

Original mean cost is $\dfrac{200}{x}$. New mean cost is $\dfrac{200}{x-4}$.

$$\frac{200}{x-4} = \frac{200}{x} + 2.5$$

$$200x = 200(x-4) + 2.5x(x-4)$$

$$200x = 200x - 800 + 2.5x^2 - 10x$$

$$2.5x^2 - 10x - 800 = 0$$

$$x^2 - 4x - 320 = 0$$

$$(x-20)(x+16) = 0$$

$$x = 20$$

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Match 4 Round 2
Algebra 1: Quadratic Equations

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1.) __________3, -3, 5, -5_______

2.) _$96x^2 - 44x - 35 = 0$__

3.) __ $2, \frac{5}{m}$____

1.)_ Find all integer values of k such that $(x+k)^2 = k^2 + kx - 2.25$ has rational solutions.

Need the discriminant to be a perfect square.

$$(x+k)^2 = k^2 + kx - 2.25$$

$$x^2 + 2kx + k^2 = k^2 + kx - 2.25$$

$$x^2 + kx + 2.25 = 0$$

Discriminant is $k^2 - 4*1*2.25 = k^2 - 9$.
This is a perfect square when k=3, -3, 5, -5

2.) Find a quadratic equation whose solutions are $-\frac{5}{12}$ and $\frac{7}{8}$.
Express your answer as $ax^2 + bx + c = 0$, where a>0 and a, b, c are relatively prime.

$$(x + \frac{5}{12})(x - \frac{7}{8}) = 0$$

$$(x+\frac{5}{12})(x-\frac{7}{8})=0$$

$$x^2+\frac{5}{12}x-\frac{7}{8}x-\frac{35}{96}=0$$

$$x^2+\frac{10}{24}x-\frac{21}{24}x-\frac{35}{96}=0$$

$$x^2-\frac{11}{24}x-\frac{35}{96}=0$$

$$96x^2-44x-35=0$$

3. Give two solutions to the quadratic equation $mx^2-2mx+4=5x-6$, one involving m and one not involving m.

$$mx^2 - 2mx + 4 = 5x - 6$$

$$mx^2 + (-2m - 5)x + 10 = 0$$

$$x = \frac{2m + 5 \pm \sqrt{(-2m - 5)^2 - 4(m)(10)}}{2m} =$$

$$\frac{2m + 5 \pm \sqrt{4m^2 + 20m + 25 - 40m}}{2m} =$$

$$\frac{2m + 5 \pm \sqrt{4m^2 - 20m + 25}}{2m} =$$

$$\frac{2m + 5 \pm \sqrt{(2m - 5)^2}}{2m} =$$

$$\frac{2m + 5 \pm (2m - 5)}{2m} =$$

$$\frac{4m}{2m}\_or\_\frac{10}{2m}$$

$$2\_or\_\frac{5}{m}$$

$$2, \frac{5}{m}$$

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Match 4 Round 3
Geometry:
Similarity

1.) ________4____________

2.) __________ $\frac{16\sqrt{10}}{3}$ ________cm_

Note: Diagrams are not
Necessarily drawn to scale

3.) _________18__________cm

1. $\overline{GH}$ is parallel to $\overline{JI}$. FI=5x+4, GI = 3x-2 , IJ=5x-8, and GH=2x-1. Find x.

   FG =FI-GI=(5x+4)-(3x-2) = 2x+6. ΔFGH is similar to ΔFIJ, so

$$\frac{FG}{FI}=\frac{GH}{JI}$$

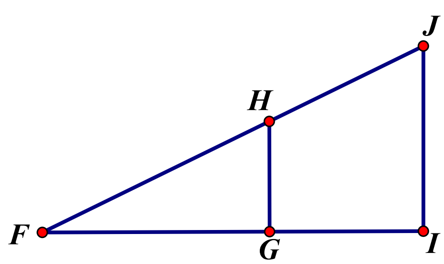
$$\frac{2x+6}{5x+4}=\frac{2x-1}{5x-8}$$

$$(5x-8)(2x+6)=(5x+4)(2x-1)$$

$$10x^2+14x-48=10x^2+3x-4$$

$$11x=44$$

$$x=4$$



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2.) The area of a regular octagon with side x is $(1+\sqrt{2})x^2$. The ratio of the sides of two regular octagons is 3:2. The area of the larger octagon is $10+10\sqrt{2}$ cm². What is the perimeter of the smaller octagon?

The ratio of the areas is the square of the ratio of the perimeters. Find the area of the smaller octagon by $\frac{9}{4}=\frac{10+10\sqrt{2}}{A}, A=\frac{40+40\sqrt{2}}{9}$. Then solve

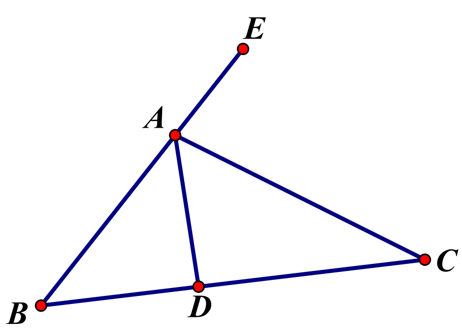
$$\frac{40+40\sqrt{2}}{9}=(1+\sqrt{2})x^2$$

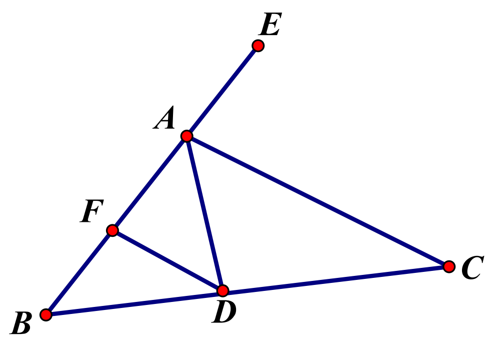
$$x^2=\frac{40+40\sqrt{2}}{9(1+\sqrt{2})}=\frac{40}{9}$$

$$x=\frac{2\sqrt{10}}{3}$$

$$Perimeter\_is\_8*\frac{2\sqrt{10}}{3}=\frac{16\sqrt{10}}{3}$$

3. In the diagram below, $\angle EAC=\angle CAD$. AB=10 cm, BD=12 cm, AD=6 cm. Find DC. (Hint: Draw a line through D parallel to $\overline{AC}$.)





Draw a segment through D parallel to $\overline{AC}$ to meet $\overline{BE}$ at F. $\angle ADF = \angle CAD$ by alternate interior angles and $\angle AFD = \angle EAC$ by corresponding angles. Since $\angle EAC = \angle CAD$, then Therefore $\angle ADF = \angle AFD$ and ΔADF is isosceles with AF=AD. Therefore

BF=BA-AF means BF=BA-AD, so BF=BA-6. BF=10-6=4. ΔBFD is similar to ΔBAC so $\frac{BD}{BC} = \frac{BF}{BA}$, .Let x=CD, so BC=12+x.

$$\frac{12}{12+x} = \frac{4}{10}$$

$$120 = 48 + 4x$$

$$4x = 72$$

$$x = 18$$

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Match 4 Round 4
Algebra 2:
Variation

1.) _____4.41_________

2.)__ $-\frac{22}{27}$ ______

3.) ___k= $\frac{1}{4}$ ______n= $\frac{1}{2}$ ____

1.) p varies directly with the square root of q, and q varies inversely with r. If p=7 when r=9, what is r when p=10?

$$p = k_1\sqrt{q}, \_ q = \frac{k_2}{r},$$

$$so\_ p = k_1\sqrt{\frac{k_2}{r}} = \frac{k_1\sqrt{k_2}}{\sqrt{r}} = \frac{K}{\sqrt{r}}$$

$$7 = \frac{K}{\sqrt{9}}\_, K = 21$$

Solve $10 = \frac{21}{\sqrt{r}}, \sqrt{r} = \frac{21}{10}, r = \frac{441}{100} = 4.41$

2.) (z+2) varies inversely with the cube of w. If z=2 when w=4, what is the value of z when w=6?

$(z+2)w^3$ is constant.

$$(2+2)*4^3 = 256, z+2 = \frac{256}{w^3}$$

$$\frac{256}{6^3} = z + 2$$

$$\frac{256}{216} = z + 2$$

$$\frac{32}{27} = z + 2$$

$$z = \frac{32}{27} - 2 = \frac{32 - 54}{27} = \frac{-22}{27}$$

3.) The ordered pair (64,2) belongs to the function $y = kx^n$ and the ordered pair (64,8) belongs to the function $y = k^3x^{n+1}$. If k is positive, find the values of k and n.

$$2 = k(64)^n$$

$$8 = k^3(64)^{n+1}$$

Divide the two equations.

$$\frac{8 = k^3(64)^{n+1}}{2 = k(64)^n}, 4 = k^2(64), k^2 = \frac{1}{16}, k = \frac{1}{4}$$

$$2 = (\frac{1}{4})(64)^n, 8 = 64^n, n = \frac{1}{2}$$

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Match 4 Round 5
Trig Expressions
and DeMoivre's
Theorem

1.) _________ 8i___________

2.) __$\sec^2 \varnothing$ or $1*\sec^2(\varnothing)$___

3.) ____________128_________

1.) Express the complex number $(2(\cos(\frac{5p}{6})+i\sin(\frac{5p}{6})))^3$ in the form a+bi.

$$(2(\cos(\frac{5p}{6})+i\sin(\frac{5p}{6})))^3 =$$

$$2^3(\cos\frac{15p}{6}+i\sin\frac{15p}{6})$$

$$=8(\cos\frac{5p}{2}+i\sin\frac{5p}{2})=8(0+i)=8i$$

2.) Express the following in the form $a[f(x)]^n$ where a and n are constants, n>0, and f(x) is a trig function of x.

$$\frac{(1+\csc ø)(1-\csc ø)(1-\sec ø)(1+\sec ø+1)}{(1+\sin ø)(1-\sin ø)}$$

$$=\frac{(1-\csc^2 ø)(1-\sec^2 ø)}{(1-\sin^2 ø)}=$$

$$\frac{(-\cot^2 ø)(-\tan^2 ø)}{\cos^2 ø}=\frac{-\frac{\cos^2 ø}{\sin^2 ø}(-\frac{\sin^2 ø}{\cos^2 ø})}{\cos^2 ø}=\frac{1}{\cos^2 ø}=\sec^2 ø$$

3. If cos(8ø) is expressed in terms of cos(ø), what is the coefficient of $\cos^8$ (ø)?

$$\cos(8ø)=\cos(2(4ø))=2\cos^2(4ø)-1=$$

$$2\cos^2(2(2ø))-1=2[2\cos^2(2ø)-1]^2\ -1$$

$$=2\{2[2\cos^2(x)-1]^2\ -1\}^2\ -1=$$

$$2\{2[4\cos^4(x)-4\cos^2(x)+1]-1\}^2\ =$$

$$2\{8\cos^4(x)-8\cos^2(x)+1\}^2$$

When the expression in the braces is squared, there will be a term of $64\cos^8(x)$, multiply the 64 by 2 to get 128.

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Match 4 Round 6
Conics

1.) __(3,1.2)______________

2.)__ (3,4)_

3.)___ $\dfrac{\sqrt{11730}}{15}$ _________

1.) A parabola has vertex (3,1) and opens upward. It passes through the point (5,6). Give the location of the focus of the parabola as an ordered pair.

$$y-1=\frac{1}{4p}(x-3)^2$$

$$6-1=\frac{1}{4p}(5-3)^2$$

$$5=\frac{1}{p},\ p=\frac{1}{5}$$

Add $\frac{1}{5}$ to the y-coordinate of the vertex to get (3,1.2)

2.) A circle and a hyperbola are centered at the origin. The hyperbola has equation $\frac{x^2}{9}-\frac{y^2}{16}=1$. The circle passes through the two foci of the hyperbola. What are the coordinates of the point in the first quadrant where the circle intersects one of the asymptotes of the hyperbola?

The foci of the hyperbola are at (5,0) and (-5,0), since $5=\sqrt{9+16}$. The radius of the circle is 5. The asymptote of the hyperbola passing through the first quadrant is $y=\frac{4}{3}x$.

$$x^2+(\frac{4}{3}x)^2=25$$

$$x^2+\frac{16}{9}x^2=25$$

$$\frac{25}{9}x^2=25, x=3, y=4$$

3.) An ellipse is centered at (0,0) and passes through the points (3,4) and (5,1). If the foci are on one of the axes, give the sum of the distances from a point on the ellipse to each of the foci.

We have the system $\begin{matrix}\frac{9}{a^2}+\frac{16}{b^2}=2\\ \frac{25}{a^2}+\frac{1}{b^2}=2\end{matrix}$,

$$\frac{9}{a^2}+\frac{16}{b^2}=2$$

$$-\frac{400}{a^2}-\frac{16}{b^2}=-32$$

Add the two equations to get $\frac{-391}{a^2}=-30, a^2=\frac{391}{30}$.

$$\frac{25}{\frac{391}{30}}+\frac{1}{b^2}=2$$

$$\frac{750}{391}+\frac{1}{b^2}=2, \frac{1}{b^2}=\frac{782}{391}-\frac{750}{391}=\frac{32}{391}, b^2=\frac{391}{32}$$

$a^2>b^2$, so the foci are on the x-axis and the sum of the distances is 2a.

$391=20^2-9^2=(20+9)(20-9)=29*11$ so 391 has no common factors with 30, so the radical will not simplify when the denominator is rationalized.

$$2a = 2\sqrt{\frac{391}{30}} = 2\frac{\sqrt{11730}}{30} = \frac{\sqrt{11730}}{15}$$

FAIRFIELD COUNTY MATH LEAGUE 2017-2018 Match 4 Team Round

1.) ______79_____________________

4.) _________ $\frac{-30\sqrt{6} - 73}{77}$ ___

2.) _____ $54+18\sqrt{3}$, $18+6\sqrt{3}$ ___

5.) _________ $\frac{40}{3}$ ___________

3.) ______(3,-2), (15,8), (15,-4)__

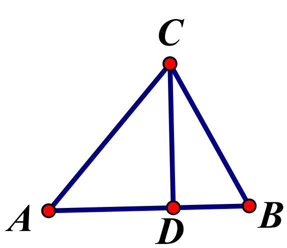
6.) _________ $\frac{41}{400}$ ___________

1. There is a set of 5 consecutive prime numbers, all of which are less than 100, for which the mean of the numbers is equal to the median of the numbers. Give the median of these numbers.

Try consecutive prime numbers less than 100, and the only one that works is

$$\frac{71+73+79+83+89}{5} = \frac{395}{5} = 79 \quad \frac{71+73+79+83+89}{5} = \frac{395}{5} = 79$$

2.)_ In the diagram below, not necessarily drawn to scale, ΔABC is a right triangle. The altitude from C is drawn to $\overline{AB}$ and intersects $\overline{AB}$ at D . AD=4x-9, BD=9, and CD= $x\sqrt{3}$. Give all possible values for the perimeter of ΔABC.



ΔADC is similar to ΔCDB, so

$$\frac{AD}{CD} = \frac{CD}{DB}$$

$$\frac{4x - 9}{x\sqrt{3}} = \frac{x\sqrt{3}}{9}$$

$$(4x - 9)9 = 3x^2$$

$$36x - 81 = 3x^2$$

$$3x^2 - 36x + 81 = 0$$

$$x^2 - 12x + 27 = 0$$

$$(x - 9)(x - 3) = 0$$

$$x = 9, x = 3$$

$$9\sqrt{3}$$

$$18\sqrt{3}$$

$$36 + 18 + 18\sqrt{3} = 54 + 18\sqrt{3}$$

$$3\sqrt{3}$$

$$CB = 6\sqrt{3}$$

$$12 + 6 + 6\sqrt{3} = 18 + 6\sqrt{3}$$

If x=9, AD=27, CD= $9\sqrt{3}$, by 30-60-90 triangles, CB=18, and AC= $18\sqrt{3}$, perimeter is $36 + 18 + 18\sqrt{3} = 54 + 18\sqrt{3}$

If x=3, AD=3, CD= $3\sqrt{3}$, AC=6 by 30-60-90 triangles, $CB = 6\sqrt{3}$
AB=12, so perimeter is $12 + 6 + 6\sqrt{3} = 18 + 6\sqrt{3}$.

3.) A parabola has focus at (-2.5, 2) and its directrix is the line x= -3.5. Find all points of intersection with the parabola and the circle with equation $x^2 - 14x + y^2 - 4y = 47$.

The parabola must open sideways in order to have its directrix be vertical, and opens to the right since -2.5 >-3.5. The x-coordinate of the vertex must be halfway between -2.5 and -3.5,and its y-coordinate must be 2 since it must have the same y-coordinate as the focus, so the vertex is at (-3,2).

Since p= $\frac{1}{2}$ (distance between focus and vertex), its equation is

$$x+3=\frac{1}{4(\frac{1}{2})}(y-2)^2\_or\_x+3=\frac{1}{2}(y-2)^2.$$

Complete the square to find the center and radius of the circle

$$x^2-14x+y^2-4y=47$$

$$x^2-14x+49+y^2-4y+4=49+4+47$$

$$(x-7)^2+(y-2)^2=100$$

so the circle is centered at (7,2) with radius 10.

If $x+3=\frac{1}{2}(y-2)^2$, then $2x+6=(y-2)^2$, so

$$(x-7)^2+(y-2)^2=100$$

$$(x-7)^2+2x+6=100$$

$$x^2-14x+49+2x+6=100$$

$$x^2-12x-45=0$$

$$(x-15)(x+3)=0$$

$$x=15\_or\_x=-3$$

If x=-3, y=2 since that is the vertex of the parabola.   If x=15,

$$2x+6=(y-2)^2$$

$$2(15)+6=(y-2)^2$$

$$36=(y-2)^2$$

$$y-2=\pm 6$$

$$y=8\_or\_y=-4$$

Points of intersection are (3,-2), (15,8), (15,-4)

4.) A is an angle in the first quadrant and B is an angle in the second quadrant. $\sec(A) = \frac{7}{5}$ and $\csc(B) = \frac{11}{5}$. Find $\cos(A+B) - \sin(A-B)$.

$\cos(A+B) - \sin(A-B)$

$\sec(A) = \frac{7}{5}, \cos(A) = \frac{5}{7}, \sin(A) = \frac{2\sqrt{6}}{7}$

$\csc(B) = \frac{11}{5}, \sin(B) = \frac{5}{11}, \cos(B) = \frac{-4\sqrt{6}}{11}$

$\cos(A+B) = \cos A \cos B - \sin A \sin B =$

$\frac{5}{7} * \frac{-4\sqrt{6}}{11} - \frac{2\sqrt{6}}{7} * (\frac{5}{11}) = \frac{-20\sqrt{6}}{77} - \frac{10\sqrt{6}}{77} = \frac{-30\sqrt{6}}{77}$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$= \frac{2\sqrt{6}}{7} * (\frac{-4\sqrt{6}}{11}) - \frac{5}{7} * \frac{5}{11} = \frac{-48}{77} - \frac{25}{77} = \frac{-73}{77}$

$Answer\_is\_ \frac{-30\sqrt{6} - 73}{77}$

5.) The acceleration due to gravity on the surface of a spherical planet is directly proportional to its mass and inversely proportional to the square of its radius. The density of an object is its mass divided by its volume. The volume of a sphere is directly proportional to the cube of its radius. Suppose acceleration due to gravity of earth is 10 meters/second^2. What is the acceleration due to gravity in meters/second^2 on the surface of a spherical planet that has twice the radius of the earth and its density is two-thirds of the density of the earth?

Let g=acceleration due to gravity, M = mass, R= radius, V=volume. M_E= mass of earth, R_E=radius of earth. Then $10 = k_1 \frac{M_E}{R_E^2}$ If the only difference in the two planets were the radii, $g = k_1 \frac{M}{R^2}$ means the acceleration due to gravity would be one-fourth that of the earth. Doubling the radius means the planet has 8 times the volume of the earth since

$V = k_2 R^3$. But since the density is two-thirds that of the earth, the mass of the planet must be $\frac{16}{3}$ the mass of the earth, since $\frac{\frac{16}{3}M_E}{8V_E} = \frac{2}{3}$.   The acceleration due to gravity on the planet must be

$$k_1 \frac{\frac{16}{3}M_E}{(2R_E)^2} = \frac{4}{3}(k_1 \frac{M_E}{R_E^2}) = \frac{4}{3}(10) = \frac{40}{3}$$

6.) An ellipse with equation $Px^2 + Qy^2 = 1$ has foci at (3,0) and (-3,0) and passes through through the point $(\frac{5\sqrt{5}}{3}, \frac{8}{3})$. Find the sum P+Q.

Since the foci of the ellipse are (3,0) an (-3, 0) it is centered at the origin and can be put into the form $\frac{x^2}{a^2} + \frac{y^2}{a^2 - 9} = 1$.

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - 9} = 1$$

$$\frac{\frac{125}{9}}{a^2} + \frac{\frac{64}{9}}{a^2 - 9} = 1$$

$$\frac{125}{9a^2} + \frac{64}{9(a^2 - 9)} = 1$$

$$125(a^2 - 9) + 64a^2 = 9a^2(a^2 - 9)$$

$$125a^2 - 1125 + 64a^2 = 9a^4 - 81a^2$$

$$9a^4 - 270a^2 + 1125 = 0$$

$$a^4 - 30a^2 + 125 = 0$$

$$(a^2 - 5)(a^2 - 25) = 0$$

$$a = 5\_or\_a = \sqrt{5}$$

But if $a = \sqrt{5}$, the intercept of the ellipse would be inside the focus, so a=5.

a^2 -9=16, so the ellipse has form $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

$$\frac{1}{25} + \frac{1}{16} = \frac{16}{400} + \frac{25}{400} = \frac{41}{400}$$