FAIRFIELD COUNTY MATH LEAGUE 2016-2017

Match 4 Round 1 Arithmetic: Basic Statistics

1.) _________8_________

2.) ________ $\frac{251}{3}$ _____________

3.) _________79____________

1.) Find the positive difference between the arithmetic mean and the median of the ten smallest positive perfect squares.

Squares are 1,4,9,16,25,36,49,64,81,100.

Median is $\frac{25+36}{2}=30.5$. Mean is

$$\frac{1+4+9+16+25+36+49+64+81+100}{10}=38.5$$

Difference is 8.

2.) The geometric mean of the numbers $x_1, x_2, x_3, \ldots x_n$ is defined to be $\sqrt[n]{x_1 x_2 x_3 \ldots x_n}$. First find the geometric mean G of the set of numbers {5,7,35,175,245}. Give as your answer the arithmetic mean of the six numbers {5,7,35,175,245,G}

To find G, putting these into prime factors, we get, $5,7,5*7,5^2*7, 5*7^2 = 5^5 7^5$

$\sqrt[5]{5^5 7^5} = 5*7 = 35$. Then $\frac{5+7+35+175+225+35}{6} = \frac{502}{6} = \frac{251}{3}$

3.)_For an ordered set of 5 distinct numbers, the lower quartile is halfway between the first and second numbers and the upper quartile is halfway between the fourth and fifth numbers. There is a set of 5 consecutive prime numbers, all of which are less than 100, such that the difference between the upper quartile and the median is equal to the difference between the median and the lower quartile. Give the median of this set.

The only set is {71,73,79,83,89}. The lower quartile is $\frac{71+73}{2}=72$. The upper quartile is $\frac{83+89}{2}=86$. 86-79 = 79-72 = 7 . The median of the set is 79.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2016-2017

Match 4 Round 2
Algebra 1:
Quadratic
Equations

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1.) __________ $4, \frac{2}{3}$__________

2.) _____________12_________

3.) __ $25x^2 - 60x + 9 = 0$____

1.)_Find all solutions to the equation 2x(x-3)=(0.5x-1)(x+4).

$$2x(x-3) = (0.5x-1)(x+4)$$
$$2x^2 - 6x = 0.5x^2 + x - 4$$
$$1.5x^2 - 7x + 4 = 0$$
$$3x^2 - 14x + 8 = 0$$
$$(x-4)(3x-2) = 0$$
$$x = 4, x = \frac{2}{3}$$

2.) For how many integer values of k does the equation $4x^2 + kx - 15 = 0$ have two rational solutions?

This becomes all the factors of 60 and their opposites in order for k to be an integer.

(4x+1)(x-15), k=-59	(4x+5)(x-3), k=-7	(2x+1)(2x-15), k=-28
(4x+15)(x-1), k=11	(4x-5)(x+3), k=7	(2x-1)(2x+15), k=28
(4x-1)(x+15), k=59	(4x+3)(x-5), k=-17	(2x-3)(2x+5), k=4
(4x-15)(x+1), k=-11	(4x-3)(x+5), k=17	(2x+5)(2x-3), k=-4

so there are 12.

3.)_Find a quadratic equation whose solutions are $\frac{6 \pm 3\sqrt{3}}{5}$.

Express your answer as $ax^2 + bx + c = 0$, where a>0 and a, b, c are relatively prime.

$Sum\_is\_\frac{-b}{a}.$

$\frac{6+3\sqrt{3}}{5}+\frac{6-3\sqrt{3}}{5}=\frac{12}{5}=\frac{60}{25}$

$\Pr oduct\_is\_\frac{c}{a}$

$(\frac{6+3\sqrt{3}}{5})(\frac{6-3\sqrt{3}}{5})=\frac{36-27}{25}=\frac{9}{25}$

$Let\_a=25.$

$Then\_b=-60, c=9$

$25x^2 – 60x + 9 = 0$

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FAIRFIELD COUNTY MATH LEAGUE 2016-2017

Match 4 Round 3
Geometry:
Similarity

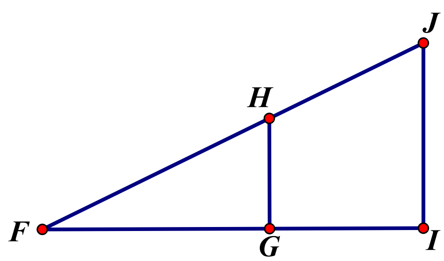
1.) ________12___________feet__

2.) __________ $12\sqrt{2}$ ________cm_

Note: Diagrams are not
Necessarily drawn to scale

3.) _________ $\frac{18\sqrt{13}}{13}$ __________cm

1. From point F, the angle of elevation to the tops of two trees shown by $\overline{GH}$ and $\overline{IJ}$ as shown below is the same. The height of the tree represented by $\overline{GH}$ is 20 feet and the height of the tree represented by $\overline{IJ}$ is 30 feet. The distance GI between the two trees is 6 feet. $\angle FGH$ and $\angle FIJ$ are right angles. Find FG.



ΔFGH is similar to ΔFIJ. $\frac{FG}{FI}=\frac{GH}{IJ}, \frac{FG}{FG+6}=\frac{20}{30}, 30FG=20(FG+6)$

$10FG=120, FG=12$

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2.) Regular hexagon ABCDEF is such that the numerical value of its perimeter in cm is equal to the numerical value of its area in cm^2. The ratio of the areas of ABCDEF to regular hexagon UVWXYZ is $\frac{2}{3}$. Find the perimeter of UVWXYZ in cm.

If s is the length of a side of ABCDEF, then

$$6s = 6*\frac{s^2\sqrt{3}}{4}, 1 = \frac{s\sqrt{3}}{4}, s = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

The area of ABCDEF is $6*\frac{(\frac{4\sqrt{3}}{3})^2\sqrt{3}}{4} = 6*\frac{\frac{48}{9}*\sqrt{3}}{4} = \frac{6*48}{9*4}\sqrt{3} = 8\sqrt{3},$

which is the same as $6*\frac{4\sqrt{3}}{3}$, so this makes sense.

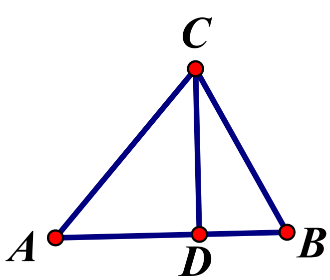
If A is the area of UVWXYZ, then $\frac{8\sqrt{3}}{A} = \frac{2}{3}, A = 12\sqrt{3}.$

To find the length of a side x of UVWXYZ,

$$\frac{x^2\sqrt{3}}{4} = 12\sqrt{3},\_so\_x^2 = 8,\_so\_x = 2\sqrt{2}$$

The perimeter of UVWXYZ is $12\sqrt{2}$.

3. In the diagram below, ΔABC is a right triangle. The altitude from C is drawn to $\overline{AB}$ and intersects $\overline{AB}$ at D. AC=6 cm and BC=4 cm. Find AD



$AB = \sqrt{4^2 + 6^2} = 2\sqrt{13}$ ΔACD is similar to ΔABC so

$$\frac{AC}{AB} = \frac{CD}{BC},$$

$$\frac{6}{2\sqrt{13}} = \frac{CD}{4}, CD = \frac{24}{2\sqrt{13}} = \frac{12}{\sqrt{13}} = \frac{12\sqrt{13}}{13}$$

$$AD = \sqrt{6^2 - (\frac{12\sqrt{13}}{13})^2} = \sqrt{36 - \frac{144*13}{169}} =$$

$$\sqrt{36 - \frac{144}{13}} = \sqrt{\frac{36*13 - 144}{13}} = \sqrt{\frac{324}{13}} = \frac{18\sqrt{13}}{13}$$

$\frac{18\sqrt{13}}{13}$ cm.

FAIRFIELD COUNTY MATH LEAGUE 2016-2017

Match 4 Round 4
Algebra 2:
Variation

1.) ______46____________

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2.)__k=__ $\frac{1}{64}$_ ______ n=__5_____

3.) _________22___________

1.) (z+2) varies directly with the square of (y+4). If z=10 when y=8, what is the value of z when y=20?

$\frac{z+2}{(y+4)^2}$ is constant.

$$\frac{10+2}{(8+4)^2}=\frac{z+2}{(20+4)^2}$$

$$z+2=\frac{12*24^2}{12^2}=12*2^2=48$$

$$z=46$$

2. The ordered pair (4,16) belongs to the function $y=kx^n$ and the ordered pair (2,4) belongs to the function $y=kx^{n+3}$. Find the values of k and n.

$$16=k(4)^n$$

$$4=k(2)^{n+3},\_so\_4=k(2^n)*8,$$

$$\_so\_\frac{1}{2}k(2^n)$$

Divide the two equations. $\frac{16=k(4)^n}{\frac{1}{2}k(2^n)}\_so\_32=2^n, n=5$

$$16 = k(4)^5, k = \frac{4^2}{4^5} = \frac{1}{4^3} = \frac{1}{64}$$

3._ The acceleration due to gravity on a planet's surface is directly proportional to the planet's mass and inversely proportional to the square of the planet's radius. In the British system, earth has a mass of about 4 x 10^{23} slugs and a radius of about 20 million feet, and the acceleration due to gravity on its surface is $32\frac{feet}{\sec ond^2}$. The acceleration due to gravity on the surface of Mars is $12\ \frac{feet}{\sec ond^2}$, and the radius of Mars is 11 million feet. What is the exponent when the mass of Mars in slugs is written in scientific notation?

Let g=acceleration due to gravity, G= constant of proportionality, M=mass of planet, and R=radius of planet.

$$g = \frac{GM}{R^2}$$

$$32 = \frac{G(4x10^{23})}{(2x10^7)^2}$$

$$32 = \frac{G(4x10^{23})}{4x10^{14}}$$

$$G = \frac{32}{10^9}$$

$$12 = \frac{\frac{32}{10^9}(M)}{(1.1x10^7)^2}$$

$$12 = \frac{\frac{32}{10^9}(M)}{(1.21x10^{14})}$$

$$14.52x10^{14} = \frac{32}{10^9}M$$

$$M = \frac{1.452x10^1 x10^{14} x10^9}{32} \approx$$

$$(4.5x10^{-2})(10^1)(10^{14})(10^9)$$

$$\approx 4.5x10^{22}$$

Answer is 22

FAIRFIELD COUNTY MATH LEAGUE 2016-2017

Match 4 Round 5
Trig Expressions
and DeMoivre's
Theorem

1.) _________ 3___________

2.) ______ $\sqrt{2}$ cis 202.5_______

3.) ___________cos(x)_________

1.) Simplify as much as possible:

$$\cos^2(\frac{\pi}{12})+\sec^2(\frac{\pi}{12})+\sin^2(\frac{\pi}{12})+\csc^2(\frac{\pi}{12})-\tan^2(\frac{\pi}{12})-\cot^2(\frac{\pi}{12})$$

$$(\cos^2(\frac{\pi}{12})+\sin^2(\frac{\pi}{12}))+(\sec^2(\frac{\pi}{12})-\tan^2(\frac{\pi}{12}))+(\csc^2(\frac{\pi}{12})-\cot^2(\frac{\pi}{12}))$$

=1+1+1 =3

2.) Find the square root of $\sqrt{2}+i\sqrt{2}$ that is in the third quadrant. Express your answer as r cis ø, where r>0 and ø is in degrees, 180 < ø <270.

For the original number, $r=\sqrt{(\sqrt{2})^2+(\sqrt{2})^2}=2$, so its square root has $r=\sqrt{2}$.

ø for $\sqrt{2}+i\sqrt{2}$ is 45 degrees, since $\tan^{-1}(\frac{\sqrt{2}}{\sqrt{2}})=45$ degrees, so the square root has angle half of that, or 22.5 degrees. Since we want the square root in the third quadrant, add 180 degrees to get 202.5 degrees.

Express as simply as possible in terms of cos x:

2.) $2[2\cos^2(\frac{x}{4})-1][1-2\sin^2(\frac{x}{4})]-1$

$$2[2\cos^2(\frac{x}{4})-1][1-2\sin^2(\frac{x}{4})]-1$$

$$=2[\cos(\frac{2x}{4})][\cos(\frac{2x}{4})]-1$$

$$=2[\cos(\frac{x}{2})][\cos(\frac{x}{2})]-1$$

$$=\cos(2*\frac{x}{2})=\cos(x)$$

FAIRFIELD COUNTY MATH LEAGUE 2016-2017

Match 4 Round 6 Conics

1.) __1.25____________

2.)__ $(-\frac{2}{5},-\frac{4}{5}),(5,1)$_

3.)___ $8\sqrt{30}$ _________

1.) The point (10,5) is on the parabola x-9 = $(y-4)^2$. What is the distance between (10,5) and the focus of x-9 = $(y-4)^2$?

Since the parabola has form $x-h=\frac{1}{4p}(y-k)^2$, $\frac{1}{4p}=1, p=\frac{1}{4}$, so the distance between the vertex (9,4) and the directrix is $\frac{1}{4}$. Since the parabola opens to the right, the directrix has equation $x=8.75$. The perpendicular distance between (10,5) is $10-8.75=1.25$. Alternatively, you could find the focus of the parabola at (9.25,4) and use the distance formula to find the distance between (9.25,4) and (10,5).

2.) Give the intersection points of the line x-3y=2 and the circle with center (1,4) and radius 5.

$(x-1)^2+(y-4)^2=25$

$x-3y=2,\_so\_x=3y+2$

$(3y+2-1)^2+(y-4)^2=25$

$(3y+1)^2+(y-4)^2=25$

$9y^2+6y+1+y^2-8y+16=25$

$10y^2-2y-8=0$

$5y^2-y-4=0$

$(y-1)(5y+4)=0$

$y=1\_or\_y=\frac{-4}{5}$

$If\_y=1, x-3*1=2, x=5$

$If\_y=\frac{-4}{5}, x-3*\frac{-4}{5}=2, x=\frac{-2}{5}$

$(\frac{-2}{5},\frac{-4}{5}),(5,1)$

3.) An ellipse with equation $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ has its foci at the same coordinates as the x-intercepts of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{36}=1$. The asymptotes of the hyperbola have equations $y=\pm 3x$. What is the area of the rhombus whose vertices are the two focal points of the hyperbola and the two y-intercepts of the ellipse?

From the equations of the asymptotes of the hyperbola we have that $\frac{b}{a}=3,\ so\ \frac{b^2}{a^2}=9$.

Since $b^2=36,\ a^2=4$, so the x-intercepts of the hyperbola are at (2,0), and (-2,0).

The x-intercepts of the ellipse are (4,0) and (-4,0), and its foci are at $c^2 = 4^2 - 2^2 = 12$, or $(0, \pm 2\sqrt{3})$, which are also the y-intercepts of the ellipse.

The foci of the hyperbola are at $(\pm c, 0)$, where $c^2 = 36+4$, so $c^2 = 40$ and $c = \pm 2\sqrt{10}$. The area of a rhombus is half the product of the diagonals, so $\frac{1}{2}(2(2\sqrt{10})(2(2\sqrt{3}) = 8\sqrt{30}$

FAIRFIELD COUNTY MATH LEAGUE 2016-2017 Match 4 Team Round

1.) _____$5+\sqrt{25+10S}$_____ 4.) ________$\frac{1}{9}$_________

2.) _________m+3, 1____________ 5.) ________-5__________

3.) __________20________________ 6.) ________$\frac{25}{4}$__________

1. One set of nonnegative integers has sum S. Another set with 10 fewer nonnegative integers also has sum S The difference between the arithmetic means of the two sets is exactly 1. If the original set has N elements, solve for N in terms of S. Express your answer in simplest radical form.

$$\frac{S}{N-10}-\frac{S}{N}=1, NS-(N-10)S=N(N-10),$$
$$10S=N(N-10), N^2-10N-10S=0$$
$$N=\frac{10\pm\sqrt{100+40S}}{2}=\frac{10\pm 2\sqrt{25+10S}}{2}=$$
$$=5\pm\sqrt{25+10S}$$

Use the positive square root, since there is no integer value of S for which 25+10S is a perfect square that is less than 5 (except S=0 and N=0, and we know the set contains at least one positive number in order for the difference in arithmetic means to be nonzero) and the sum must be positive (also N must be greater than or equal to 10) so
$5+\sqrt{25+10S}$

2.): The quadratic equation $x^2-(m+4)x+(m+3)=0$ has two solutions, one that is a real number with no variable and the other an expression involving m. Give both solutions.

By quadratic formula,

$$x = \frac{(m+4) \pm \sqrt{(m+4)^2 - 4(m+3)}}{2} =$$

$$\frac{(m+4) \pm \sqrt{(m^2 + 8m + 16 - 4m - 12)}}{2}$$

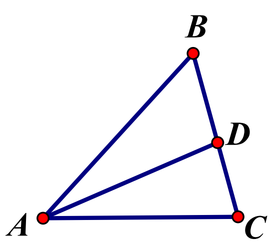
$$= \frac{(m+4) \pm \sqrt{(m^2 + 8m + 16 - 4m - 12)}}{2}$$

$$= \frac{(m+4) \pm \sqrt{m^2 + 4m + 4}}{-2}$$

$$= \frac{(m+4) \pm (m+2)}{2}$$

$$= \frac{(m+4) + (m+2)}{2} \_or\_ \frac{(m+4) - (m+2)}{2}$$

$$= \frac{2m+6}{2} \_or\_ \frac{2}{2} = m+3, 1$$



3. In ΔABC, the angle bisector of $\angle BAC$ intersects $\overline{BC}$ at point D.
If AB=9x-5, BD=6x+2, BC=12x-8, and AC=4x+5, find the length of $\overline{CD}$

By the angle bisector theorem from the similar triangles unit,

$$\frac{9x-5}{4x+5}=\frac{6x+2}{(12x-8)-(6x+2)}$$

$$\frac{9x-5}{4x+5}=\frac{6x+2}{6x-10}$$

$$(9x-5)(6x-10)=(4x+5)(6x+2)$$

$$54x^2-120x+50=24x^2+38x+10$$

$$30x^2-158x+40=0$$

$$15x^2-79x+20=0$$

$$(x-5)(15x-4)=0$$

$$x=5, x=\frac{4}{15}$$

Reject $\frac{4}{15}$ because it makes one of the sides negative, so x=5, and the length of $\overline{CD}$ = 6*5-10=20

4.)_The ordered pairs $(12,\frac{1}{72})$ and $(27,\frac{1}{243})$ belong to an inverse variation function $y=\frac{k}{x^n}$ where n is a fraction. What is x when the value of y is $9\sqrt{3}$?

$$y=\frac{k}{x^n}$$

$$\frac{1}{72}=\frac{k}{12^n},\frac{1}{243}=\frac{k}{27^n}$$

$$k=\frac{12^n}{72},k=\frac{27^n}{243},(\frac{27}{12})^n=\frac{243}{72},(\frac{9}{4})^n=\frac{27}{8}$$

Since $9^{\frac{3}{2}}=27\_and\_4^{\frac{3}{2}}=8,n=\frac{3}{2}$

$$y=\frac{k}{x^n},\frac{1}{72}=\frac{k}{12^{\frac{3}{2}}},\frac{1}{72}=\frac{k}{(2\sqrt{3})^3},k=\frac{24\sqrt{3}}{72}=\frac{\sqrt{3}}{3}.$$

Solve $9\sqrt{3}=\frac{\frac{\sqrt{3}}{3}}{x^{\frac{3}{2}}},x^{\frac{3}{2}}=\frac{\sqrt{3}}{27\sqrt{3}}=\frac{1}{27},x=\frac{1}{9}$

5.) $\tan(4x)=\frac{A\tan^3(x)+B\tan(x)}{\tan^4(x)+C\tan^2(x)+D}$ for some integer values of A, B, C, and D. Find A+B+C+D

$$\tan(4x)=\frac{2\tan(2x)}{1-\tan^2(2x)}=\frac{\frac{2(2\tan x)}{1-\tan^2 x}}{(1+\tan(2x))(1-\tan(2x))}$$

$$=\frac{\frac{2(2\tan x)}{1-\tan^2 x}}{(1+\frac{2\tan x}{1-\tan^2 x})(1-\frac{2\tan x}{1-\tan^2 x})}=\frac{\frac{2(2\tan x)}{1-\tan^2 x}}{(\frac{1-\tan^2 x+2\tan x}{1-\tan^2 x})(\frac{1-\tan^2 x-2\tan x}{1-\tan^2 x})}$$

$$\frac{4\tan x}{(\frac{(1-\tan^2 x+2\tan x)(1-\tan^2 x-2\tan x)}{1-\tan^2 x})}=$$

$$\frac{4\tan x(1-\tan^2 x)}{(\frac{(1-\tan^2 x+2\tan x)(1-\tan^2 x-2\tan x)}{1-\tan^2 x})}=$$

$$\frac{4\tan x(1-\tan^2 x)}{(\tan^4 x-6\tan^2 x+1)}=\frac{-4\tan^3 x+4\tan x)}{(\tan^4 x-6\tan^2 x+1)}$$

$$4-4-6+1=-5$$

6.) A circle of radius 5 centered on the positive y-axis is tangent to both of the asymptotes of $9x^2-16y^2=144$. Give the y-coordinate of the center of the circle.

$$9x^2 - 16y^2 = 144$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Asymptotes are $y = \frac{3}{4}x, y = -\frac{3}{4}x$, so the points on the circle must be

$$(x, \frac{3}{4}x)\_and\_(x, -\frac{3}{4}x).$$

The circle is tangent to the asymptotes at these points, so the lines from the center of the circle to these points must have slopes $\frac{-4}{3}\_and\_\frac{4}{3}$. Let k be the y-coordinate of the center of the circle. Then

$$\frac{3}{4}x - k = \frac{-4}{3}(x - 0), \frac{-3}{4}x - k = \frac{4}{3}(x - 0)$$

$$k = \pm\frac{25}{12}x$$

$$(x - 0)^2 + (\frac{3}{4}x - \frac{25}{12}x)^2 = 25$$

$$x^2 + (\frac{-4}{3}x)^2 = 25$$

$$\frac{25}{9}x^2 = 25, x = \pm 3, y = \pm\frac{9}{4}$$

Since we are looking for the value of k on the positive y-axis, the positive value of y when x = ± 3,

$$k = \frac{25}{12} * 3 = \frac{25}{4}$$