FAIRFIELD COUNTY MATH LEAGUE (FCML) 2014-2015

Match 4 Round 1
Arithmetic:
Basic Statistics

1.) _________0.2_________

2.) ________32________

3.) _______$8\sqrt{2}$__________

1.) Let S={the 10 smallest natural numbers that are neither multiples of 3 nor multiples of 5}. Find the positive difference between the arithmetic mean and the median of this set.

S consists of the set {1,2,4,7,8,11,13,14,16,17}. The median is halfway between 8 and 11, so 9.5. The mean is 93/10, so 9.5-9.3=0.2

2.)_A set of N Math SAT scores has arithmetic mean 550. When you remove four scores of 520, 540, 660, and 620, the arithmetic mean drops to 545. Find the value of N.

The sum of the scores is N*550. The sum of the scores is also (N-4)*545+520+540+660+620

So 550N=545N-2180+2340

5N=160, and N=32.

3.) The geometric mean of the numbers $x_1, x_2, x_3, \ldots x_n$ is defined to be $\sqrt[n]{x_1 x_2 x_3 \ldots x_n}$ If T is the set of positive natural number factors of 128, what is the geometric mean of the elements of T? Express your answer in the form in simplest radical form $a\sqrt[c]{b}$

T is the set {1,2,4,8,16,32,64, 128}=(2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7}

Multiplying these gives 2^{28}. Taking the eighth root is

$$\sqrt[8]{2^{28}} = \sqrt[8]{2^{24}}\sqrt[8]{2^4} = 2^3 * 2^{\frac{4}{8}} = 8 * 2^{\frac{1}{2}} = 8\sqrt{2}$$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 4 Round 2
Algebra 1:
Quadratic
Equations

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1.) ________0.5, 2.5____________

2.) ___________ $\frac{1}{2}$ *or* $\frac{1}{5k-2}$ ________

3.) ________ 15, 20, 24____________

1.) Find all solutions to the equation $(x-3.5)^2=(2x-4)^2$

$x^2-7x+12.25 = 4x^2-16x+16$

$0=3x^2-9x+3.75$

$0=12x^2-36x+15$

$0=3(2x-1)(2x-5)$

$x=0.5$ or $x=2.5$

2.) If $k\neq 0.4$, give all solutions to the quadratic equation $(10k-4)x^2 - 5kx + 1 = 0$ in terms of k.

$$x = \frac{5k \pm \sqrt{(-5k)^2 - 4(10k-4)}}{2(10k-4)}$$

$$= \frac{5k \pm \sqrt{25k^2 - 40k + 16}}{2(10k-4)}$$

$$= \frac{5k \pm (5k-4)}{2(10k-4)}$$

$$= \frac{5k + (5k-4)}{2(10k-4)} \text{ or } \frac{5k - (5k-4)}{2(10k-4)}$$

$$= \frac{10k-4}{2(10k-4)} \text{ or } \frac{4}{2(10k-4)}$$

$$= \frac{1}{2} \text{ or } \frac{1}{5k-2}$$

3.) Find all positive integer values of m such that the equation $\frac{m}{4}x^2 - 25x + m = 0$ has two rational solutions that are between -10 and 10.

Since the solutions must be rational, we need $25^2 - 4(\frac{m}{4})m$ to be a perfect square, so 25 must be the third number of a Pythagorean triple. We have 15-20-25 and 7-24-25, so m could be 7, 15, 20, 24.

$$\frac{25 \pm \sqrt{625 - m^2}}{2 * \frac{m}{4}} = \frac{50 \pm 2\sqrt{625 - m^2}}{m}$$

$$Substituting\_these\_4\_numbers\_for\_m,$$

$$m = 7 : \frac{50 \pm 2\sqrt{576}}{7} = \frac{94}{7}, \frac{2}{7}\_toobig$$

$$m = 15 : \frac{50 \pm 2\sqrt{400}}{15} = 6, \frac{4}{3}, OK$$

$$m = 20 : \frac{50 \pm 2\sqrt{225}}{20} = 4, 1, OK$$

$$m = 24 : \frac{50 \pm 2\sqrt{49}}{24} = \frac{64}{24}, \frac{36}{24}, OK$$

Answers are 15, 20 and 24.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2014-2015

Match 4 Round 3
Geometry:
Similarity

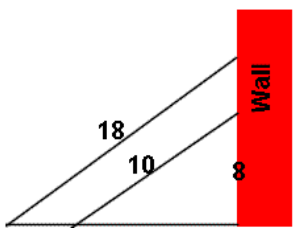
1.) ________6.4_________feet___

2.) _______$150\sqrt{3}$___________cm²__

Note: Diagrams are not
Necessarily drawn to scale

3.) OA'=____ 15____A'B'=___ $\frac{15}{4}$_____

1.) Two ladders are leaned against a wall such that they make the same angle with the ground. The 10' ladder reaches 8' up the wall. How much further up the wall does the 18' ladder reach? The wall is perpendicular to the ground.



Solution to #1: If x is the height on the wall that the 18 foot ladder reaches, $\frac{8}{10} = \frac{x}{18}$, so
10x=144,
X=14.4, and it reaches an additional
6.4 feet above the point hit by the 10 foot ladder.

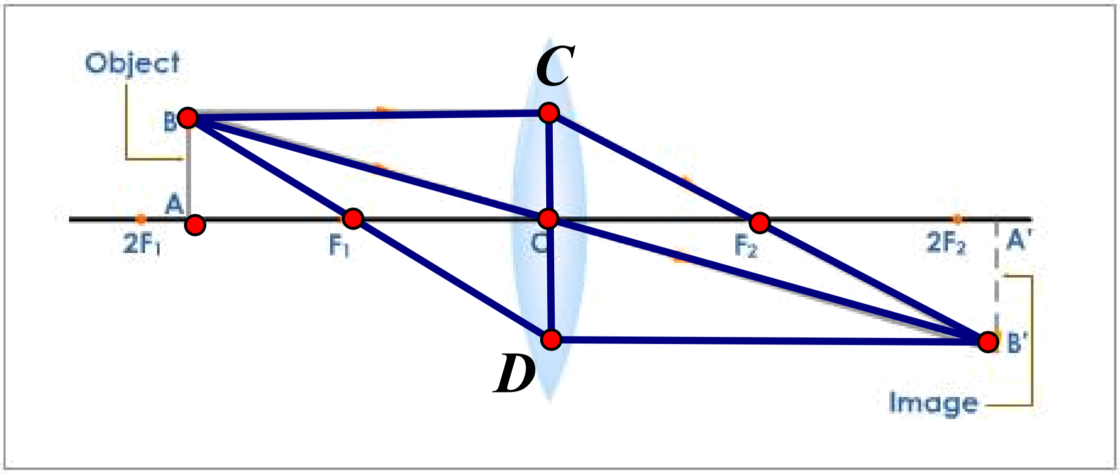
2. Two regular polygons are similar to each other. The area of the smaller polygon is $24\sqrt{3}$ cm^2. The ratio of their perimeters is 5:2. What is the area of the larger polygon in cm^2?

The ratio of the areas is the square of the ratio of the perimeters, so $\frac{25}{4} = \frac{x}{24\sqrt{3}}$

$\frac{25}{4} = \frac{x}{24\sqrt{3}}$,

So $4x = 600\sqrt{3}$

$x = 150\sqrt{3}$



3.) The picture above shows a ray diagram for a converging lens. $\overline{BC}, \overline{AA'}$, and $\overline{DB'}$ are all parallel to each other. $\overline{AB}, \overline{CD}$, and $\overline{A'B'}$ are all parallel to each other. F_1 lies on $\overline{BD}$, O lies on $\overline{CD}$, F_2 lies on $\overline{CB'}$, and F_1, O and F_2 all lie on $\overline{AA'}$. $\overline{AA'}$ is perpendicular to $\overline{AB}$. If $OF_1 = 6$, $OA = 10$, and $AB = 2.5$, find the lengths of $\overline{OA'}$ and $\overline{A'B'}$.

ΔOF_1D is similar to ΔAF_1B by having vertical angles and two right angles, so $\frac{OF_1}{AF_1} = \frac{OD}{AB}, \frac{4}{6} = \frac{2.5}{OD}, \_so\_OD = \frac{15}{4}$, and this is the same length as A'B'.

Also, ΔOAB is similar to $\Delta OA'B'$ by vertical angles and right angles, so

$$\frac{OA}{OA'} = \frac{AB}{A'B'}, \frac{10}{OA'} = \frac{2.5}{\frac{15}{4}}, \_so\_OA' = 15.$$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2014-2015

Match 4 Round 4
Algebra 2:
Variation

1.) ______0.75_____________

2.) ____________12__________mm

3.) ____________ $\frac{64}{27}$ ______________

1.) z varies inversely with the cube of (y+2). If z=6 when y=3, what is the value of z when y=8?

$Z(y+2)^3$ is constant. 6(3+2)^3 = z(8+2)^3. 6*125=z(1000) z=750/1000 =0.75

2._ The resistance in ohms of a cylindrical electrical circuit element made from a certain material is directly proportional to its length and inversely proportional to its cross sectional area. If a resistor of length 6 mm has a radius of 4 mm and a resistance of 10 ohms, what is the length in mm of a resistor with resistance of 5 ohms and radius 8 mm?

Let R=resistance, L=length=, r=radius, A=area. $A=\pi r^2$.

$$R = \frac{kL}{A} = \frac{kL}{\pi r^2}$$

$$10 = \frac{k*6}{\pi(4^2)} \backslash \quad k = \frac{160\pi}{6}.$$

$$5 = \frac{160\pi}{6} * \frac{L}{\pi * 8^2}$$

$$L = \frac{5*6*\pi*8^2}{160\pi} = 12$$

3.)z varies jointly with the cube of x and the square of y, x varies directly with $v^{\frac{1}{2}}$ and y varies inversely with $w^{\frac{2}{3}}$. If z=36 when v=9 and w=8, what is w when v=4 and z=54?

$Z=k_1x^3y^2$.

$$x = k_2 v^{\frac{1}{2}}, y = \frac{k_3}{w^{\frac{2}{3}}}.$$

$$Multiply\_k_1k_2k_3 = k,$$

$$and\_z = \frac{k9^{\frac{1}{2}}}{8^{\frac{2}{3}}}$$

$$36 = \frac{k9^{\frac{1}{2}}}{8^{\frac{2}{3}}} \backslash \quad 36 = \frac{k3}{4} \backslash \quad k = 48$$

$$54 = \frac{k4^{\frac{1}{2}}}{w^{\frac{2}{3}}} \backslash \quad 54 = \frac{96}{w^{\frac{2}{3}}} \backslash \quad w^{\frac{2}{3}} = \frac{96}{54} = \frac{16}{9} \backslash \quad w = (\frac{4}{3})^3 = \frac{64}{27}$$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2014-2015

Match 4 Round 5
Trig Expressions

1.) ________ -2____________

2.) _____ $\sec^2(x) - \csc^2(x)$________

3.) ___________32______________

1.) If x= $\frac{3\pi}{4}$, find the sum of cos(x)+sin(x)+cot(x)+tan(x)+sec(x)+csc(x).

$$\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4} + \tan\frac{3\pi}{4} + \cot\frac{3\pi}{4} + \sec\frac{3\pi}{4} + \csc\frac{3\pi}{4} =$$

$$\frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 - 1 - \sqrt{2} + \sqrt{2} = -2$$

2.) If 0<x< $\frac{\pi}{2}$, express $(\tan(x) - \cot(x))(\frac{\sec(x)+\csc(x)}{\cos(x)+\sin(x)})$ in terms of sec(x) and csc(x).

$$(\tan(x) - \cot(x))(\frac{\sec(x)+\csc(x)}{\cos(x)+\sin(x)})$$

$$(\tan(x) - \cot(x))(\frac{\frac{1}{\cos(x)} + \frac{1}{\sin(x)}}{\cos(x)+\sin(x)}) =$$

$$(\tan(x) - \cot(x))(\frac{\frac{\cos(x)+\sin(x)}{\cos(x)\sin(x)}}{\cos(x)+\sin(x)}) =$$

$$(\tan(x) - \cot(x))(\sec(x)\csc(x) =$$

$$\frac{\sin(x)}{\cos(x)\sin(x)\cos(x)} - \frac{\cos(x)}{\sin(x)\sin(x)\cos(x)}$$

$$= \sec^2(x) - \csc^2(x)$$

Factored form OK.

3.) What is the coefficient of $\cos^6(x)$ when $\cos(6x)$ is written as polynomial in $\cos(x)$?

$$\cos(3x) = \cos(2x + x) = (\cos(2x)\cos(x) - \sin(2x)\sin(x))$$

$$= (2\cos^2(x) - 1)(\cos(x)) - 2\sin^2(x)\cos(x)$$

$$= (2\cos^2(x) - 1)(\cos(x)) - 2(1 - \cos^2(x))(\cos(x))$$

$$= 4\cos^3(x) - 3\cos(x)$$

$$So\cos(6x) = 4\cos^3(2x) - 3\cos(2x) =$$

$$4(2\cos^2(x) - 1)^3 - 3(2\cos^2(x) - 1)$$

so the coefficient of $\cos^6(x)$ comes from the term where you cube $2\cos^2(x)$ and multiply by 4, so the coefficient is 32.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 4 Round 6
Conics

1.) (____1___,____-2______)

2.) ________ $\sqrt{5}$ ______________

3.)_________24_______________

1.) The circle with equation $4x^2 - 8x + 4y^2 - ky = 380$ passes through the point (7,6). Give the coordinates of the center of the circle.

$4*7^2-8*7+4*6^2-k*6=380$

196-56+144-6k=380, -6k=96, k=-16

$4x^2 - 8x + 4y^2 + 16y = 380$

$4x^2 - 8x + 4 + 4y^2 + 16y + 16 = 400$

$4(x-1)^2+4(y+2)^2=400$ Center is (1,-2)

2.) A parabola has its focus at (0,1) and its directrix is the x-axis. The parabola passes through (x,3) for some positive value of x. Find x. Express your answer in simplest radical form.

. The parabola must have its vertex at $(0,\frac{1}{2})$. Since the distance from the focus to the y-intercept is $\frac{1}{2}$, we have $\frac{1}{2}=\frac{1}{4p},\_so\_p=\frac{1}{2}$. So $y=\frac{1}{2}x^2+\frac{1}{2}$.

Find x such that $3=\frac{1}{2}x^2+\frac{1}{2}$,

$\frac{5}{2}=\frac{1}{2}x^2$, $x=\sqrt{5}$

Alternatively, we know the distance from the focus to the desired point must be 3, since the distance from (x,0) on the directrix to (x,3) is 3, so use the distance formula to solve for x when the distance is 3 and the points are (0,1) and (x,3).

3.)The foci of the two hyperbolas $5x^2$-20x-4y^2-24y=36 and $7x^2$-14x-9y^2-72y=74 are the vertices of a quadrilateral. Find the area of this quadrilateral.

$$5x^2 - 20x - 4y^2 - 24y = 36$$

$$5x^2 - 20x + 20 - 4y^2 - 24y - 36 = 20$$

$$5(x-2)^2 - 4(y+3)^2 = 20$$

$$\frac{(x-2)^2}{4} - \frac{(y+3)^2}{5} = 1$$

The foci are on the line y=-3, a distance of $\sqrt{4+5}$ from the center (2,-3), so the foci are at (-1,-3) and (5,-3)

$$7x^2 - 14x - 9y^2 - 72y = 74$$

$$7x^2 - 14x + 7 - (9y^2 + 72y + 144) = 74 + 7 - 144 = -63$$

$$7(x-1)^2 - 9(y+4)^2 = -63$$

$$\frac{(y+4)^2}{7} - \frac{(x-1)^2}{9} = 1$$

The foci are on the line x=1, a distance of $\sqrt{9+7}$ from the center (1,-4), so they are at (1,0) and (1,-8).

We now have a quadrilateral with vertices at (-1,-3), (5,-3), (1,0), and (1,-8). Draw the diagonals and they are horizontal and vertical, so they split the quadrilateral into 4 right triangles of 2 and 3, 3 and 4, 2 and 5, and 4 and 5. So the sum of the areas is
0.5*2*3+0.5*3*4+0.5*2*5+0.5*4*5=3+6+5+10=24

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2014-2015 Match 4 Team Round

1.) ______121.25___________ 4.) ________ $0, -1, \frac{3}{4}$ _________

2.) _________ $\frac{15}{23}$ ____________ 5.) ________ $80\sqrt{10}$ ___________

3.) __________ $\frac{36}{7}$ ______ 6.) ___(4, 1) , (7.75, 2.5)_____

1.)_A variation function has form $f(x)=ax^b$ for some constants a and b. When x=4, f(x)=2. When x=16, f(x)=16. The interquartile range of a set is defined as the positive difference between its upper quartile (median of the upper half of the set) and lower quartile (median of the lower half of the set). Consider the set of ten numbers {f(1), f(4), f(9), f(16)…,f(100)}. Find the interquartile range of this set.

$2 = a*4^b$ _and_ $16 = a*16^b$.

*Divi*de_*the*_*equations*_*to*_*get*

$8 = 4^b, so\_b = \frac{3}{2}$

$2 = a*4^{\frac{3}{2}}\_so\_a = \frac{1}{4}$

This is an increasing function, so the median is halfway between $f(5^2)$ and $f(6^2)$, and the interquartile range is $f(8^2)-f(3^2)=\frac{1}{4}*64^{\frac{3}{2}} - \frac{1}{4}*9^{\frac{3}{2}} = \frac{1}{4}(512 - 27) = 121.25$

$$\frac{1}{4}*64^{\frac{3}{2}} - \frac{1}{4}*9^{\frac{3}{2}} = \frac{1}{4}(512 - 27) = 121.25$$

2. In ΔABC, the angle bisector of $\angle ACB$ intersects $\overline{AB}$ at point D. If $AC=2y^2+16$, CB=y+14, AD=y+16, and DB=12, give the arithmetic mean of all possible values of y.
By the angle bisector theorem we have

$$\frac{2y^2+16}{y+14} = \frac{y+16}{12}$$

$$12(2y^2+16) = (y+14)(y+16)$$

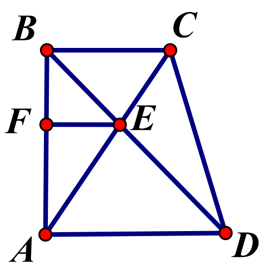
$$24y^2 + 192 = y^2 + 30y + 224$$

$$23y^2 - 30y - 32 = 0$$

$$(23y+16)(y-2) = 0$$

$$y = \frac{-16}{23}\_or\_y = 2$$

The arithmetic mean is (-16/23+2)/2= (-16/23+46/23)/2 = $\frac{15}{23}$



3. In trapezoid ABCD, the bases are $\overline{AD}$ and $\overline{BC}$. $\overline{AB}$ is perpendicular to both $\overline{AD}$ and $\overline{BC}$. Diagonals $\overline{BD}$ and $\overline{AC}$ meet at E. A line segment is drawn from E to $\overline{AB}$ parallel to $\overline{AD}$ and intersects $\overline{AB}$ at point F. If AD=12 and BC=9, find the length of $\overline{EF}$.

Let BF=x, and AB=y, so AF=y-x. ΔBEF is similar to ΔBDA so $\frac{FE}{12} = \frac{x}{y}$.

ΔAFE is similar to ΔABC, so $\frac{FE}{9} = \frac{y-x}{y} = 1 - \frac{x}{y}$, so $\frac{FE}{9} = 1 - \frac{FE}{12}$, so

4*FE=36-3*FE, so 7*FE=36, FE= $\frac{36}{7}$

4. Find all values of sin(x) for which $\sin(3x) = \sin^2(x)$.

$\sin(3x) =$

$\sin(2x + x) =$

$(\cos(2x)\sin(x) + \sin(2x)\cos(x))$

$(\cos^2(x) - \sin^2(x))(\sin(x)) + 2\sin(x)(\cos^2(x))$

$= (-\sin^3(x) + 3\sin(x)(\cos^2(x)))$

$= (-\sin^3(x) + 3\sin(x)(1 - \sin^2(x)))$

$(-4\sin^3(x) + 3\sin(x))$

$(-4\sin^3(x) + 3\sin(x))$

Set this equal to $\sin^2(x)$, and you have

$-4 \sin^3(x) + 3 \sin(x) = \sin^2(x)$

$\sin(x)(-4 \sin^2(x) - \sin x + 3) = 0$

$\sin(x)(-4 \sin(x)+3)(\sin(x)+1)$

sin(x)=0 or sin(x) = -1, or sin(x) = $\frac{3}{4}$

5. The eccentricity of an ellipse is the ratio (distance of one of its foci from the center of the ellipse):(length of semi-major axis). Kepler's third law of planetary motion states that the time it takes a planet to orbit the sun varies directly with some power of the length of the semi-major axis of its orbital path. The length of Venus' semi-major axis is 0.7 astronomical units, and the time it takes to orbit the sun in earth-years is the constant of variation multiplied by $\sqrt{0.343}$. Pluto has an eccentricity of 0.25 and the length of the semi-minor axis of its orbital path is $10\sqrt{15}$ astronomical units. The time it takes Pluto to orbit the sun in earth-years is the same constant of variation multiplied by what number? Express your answer in simplest radical form.

The expression $\sqrt{0.343}$ is intentional left not in simplest radical form so that students might recognize that $\sqrt{0.343} = \sqrt{0.7^3}$, so the exponent in the variation function is $\frac{3}{2}$.

If Pluto has an eccentricity of 0.25, then if one focus is c units from the center of the ellipse, and the endpoint of the semi-major axis is a units from the center of the ellipse, then c=(0.25a). We also have $b^2+c^2=a^2$, so $(10\sqrt{15})^2+(0.25a)^2=a^2$, so $1500=\frac{15}{16}a^2$,

So a= 40 astronomical units. The desired number is then $40^{\frac{3}{2}} = 40\sqrt{40} = 80\sqrt{10}$

6. Give the coordinates of all points where the parabola $x = y^2 - y + 4$ intersects either of the asymptotes of the hyperbola

$$4x^2 + 8x - 25y^2 - 50y = 121 =$$

$$4x^2 + 8x + 4 - 25y^2 - 50y - 25 = 121 + 4 - 25$$

$$4(x+1)^2 - 25(y+1)^2 = 100$$

$$\frac{(x+1)^2}{25} - \frac{(y+1)^2}{4} = 1$$

so the asymptotes have slopes $\pm\frac{2}{5}$ and the equations are $y+1=\pm\frac{2}{5}(x+1)$.

A drawing will show that since the parabola opens to the right, it will only intersect $y+1=\frac{2}{5}(x+1)$, so $5(y+1)=2(x+1)$. $5(y+1)=2(y^2-y+4+1)$

$5y+5=2y^2-2y+10$, $2y^2-7y+5=0$, $(y-1)(2y-5)=0$, so y=2.5 or y=1

If y=1, x=4. If y=2.5, x=7.75