FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 5 Round 1
Algebra I:
Fractions and
Exponents

1.) ________9__________

2.) ______ $\dfrac{576x^2 + 8x^3 - 9}{9x}$ _____

3.) _ $\dfrac{24yz^2 - 48z^{10} + 20y^{18} - 7y^{10}z^2}{12y^6z^8}$ __

1) Express as an integer or a reduced fraction:

   $(225)^5(75)^{-4}(45)^{-3}(15)^2(5)^{-1}(3)^0=$

   $3^{10}5^{10}5^{-8}3^{-4}5^{-3}3^{-6}5^2 3^2 5^{-1}*1$

   $= 3^2 5^0 = 9$

2) If $x = a^{12}b^{-18}c^{24}$, express as a single reduced fraction in terms of only x and constants:

$$\left(2a^2b^{-3}c^4\right)^6 + \left(a^6\right)^4\left(\frac{1}{3}b^{-18}\right)^2\left(2c^{16}\right)^3 - \left(2a^{-6}\right)^2\left(\frac{1}{4}b^6\right)^3\left(4c^{-12}\right)^2$$

$$64a^{12}b^{-18}c^{24} + \frac{8a^{24}b^{-36}c^{48}}{9} - \frac{64a^{-12}b^{18}c^{-24}}{64} =$$

$$64x + \frac{8x^2}{9} - x^{-1} = \frac{576x^2 + 8x^3 - 9}{9x}$$

3) If y≠0 and z≠0, express as a single reduced fraction with no negative exponents:

$$\frac{2}{y^5z^6} - \frac{4}{\left(y^3\right)^2 z^{-2}} + \frac{5}{3(y^{-4})^3z^8} - \frac{7}{(2y^{-2})^2(3z)(z^5)}$$

$$\frac{2}{y^5 z^6} - \frac{4z^2}{\left(y^3\right)^2} + \frac{5y^{12}}{3z^8} - \frac{7y^4}{4(3z)(z^5)} =$$

$$\frac{24yz^2}{12y^6z^8} - \frac{48z^{10}}{12y^6z^8} + \frac{20y^{18}}{12y^6z^8} - \frac{7y^{10}z^2}{12y^6z^8} =$$

$$\frac{24yz^2 - 48z^{10} + 20y^{18} - 7y^{10}z^2}{12y^6z^8}$$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 5 Round 2 Algebra I: Fractional Expressions and Equations

1.) ________ $\frac{-3}{8}$ ____________

2.) __________-2__________

3.) ________ $\frac{5x+y}{y-5x}$ _______________

1). Simplify the product as much as possible if no values of x make any denominators equal to zero:

$$\frac{x^2-10x+24}{32-2x^2}*\frac{3x^2+30x+72}{4x^2-144}=\frac{(x-6)(x-4)}{2(x+4)(4-x)}*\frac{3(x+6)(x+4)}{4(x+6)(x-6)}=\frac{-3}{8}=$$

2). Solve for all possible values of x:

$$3-\frac{22}{x+5}=\frac{6x-1}{2x+7}$$

$$3(x+5)(2x+7)-22(2x+7)=(6x-1)(x+5)$$

$$6x^2+51x+105-44x-154=6x^2+29x-5$$

$$7x-49=29x-5$$

$$-22x=44$$

$$x=-2$$

3). Simplify as much as possible given that x≠0, y≠0, 5x≠y, and 5x≠-y.

$$\frac{5xy+y^2+\dfrac{25x^2-y^2}{1-\dfrac{y}{5x}}}{y^2-5xy-\dfrac{25x^2-y^2}{1+\dfrac{y}{5x}}}=\frac{5xy+y^2+\dfrac{25x^2-y^2}{\dfrac{5x-y}{5x}}}{y^2-5xy-\dfrac{25x^2-y^2}{\dfrac{5x+y}{5x}}}=\frac{5xy+y^2+5x(5x+y)}{y^2-5xy-5x(5x-y)}$$

$$==\frac{(5x+y)(5x+y)}{(5x+y)(y-5x)}=\frac{5x+y}{y-5x}$$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

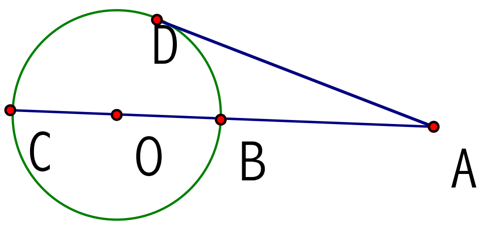
Match 5 Round 3
Geometry:
Circles

1.) ___________6_________

2.) ____________11_____________

3.) _________$4\sqrt{2} - 4$ _____________

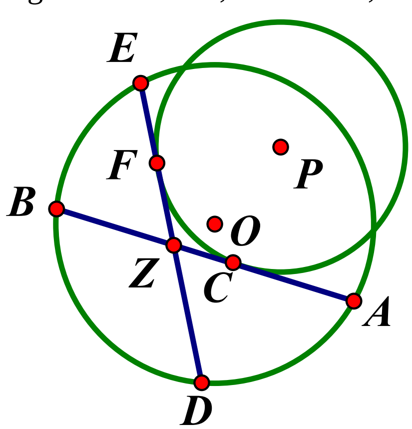
1). $\overline{BC}$ is a diameter of circle O. The line containing $\overline{BC}$ passes through point A outside the circle. $\overline{AD}$ is tangent to circle O. If AD= $\sqrt{60}$ and OC=2, find the length of $\overline{AB}$.



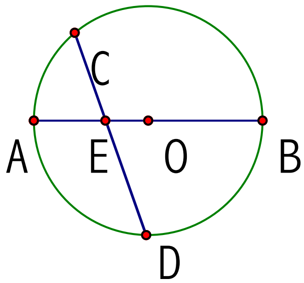
If x=length of $\overline{AB}$, $(AD)^2$=AB*AC. Since OC=2, BC=4, so $\sqrt{60}^2$=x(x+4). Solve $x^2+4x-60=0$ to get (x+10)(x-6)=0, so x=6.

2) Circle P has its center in the interior of circle O. $\overline{AB}$ and $\overline{DE}$ are chords of Circle O that meet at Z. $\overleftrightarrow{ZF}$ and $\overleftrightarrow{ZC}$ are tangent to Circle P at points F and C respectively. If AZ= $x - 5$, BZ= $\frac{x}{5}$, EZ= $x - 9$, DZ= $\frac{x}{4}$, and ZC=5, find the length of $\overline{EF}$.

AZ*BZ=EZ*DZ, so (x-5)* $\frac{x}{5}$=(x-9)* $\frac{x}{4}$. Multiply by 20 by get 4(x-5)x=5(x-9)x, so $4x^2-20x=5x^2-45x$. x can't be zero, so 4x-20=5x-45, so x=25. EZ=25-9=16. ZC and ZF are both tangent to circle P, so ZC=ZF, so ZF is also 6, and therefore EF = EZ-ZF=16-5=11.



3.) Circle O has radius 4 cm $\overline{AB}$ is a diameter of the circle and intersects chord $\overline{CD}$ at E. The length of minor arc AC is π cm and the measure of is 67.5 degrees. Find the length of $\overline{OE}$ in cm.



Since the length of arc AC is π cm, this is one-eighth of the circumference 8π of the circle so arc AC measures 45 degrees. Since angle $\angle AEC$ measures 67.5 degrees, this is one half the sum of arcs AC and BD, so arc BD must be 90 degrees.
Set the center of the circle to be at the origin. Then since arc AC is 45 degrees, C must be the point $(-2\sqrt{2}, 2\sqrt{2})$ and D must be the point (0,-4). The slope of the line connecting them is $\frac{-4-2\sqrt{2}}{2\sqrt{2}}$, and the equation of the line connecting them is y+4= $\frac{-4-2\sqrt{2}}{2\sqrt{2}}$ (x). Find x when y=0. X=
$\frac{8\sqrt{2}}{-4-2\sqrt{2}} = \frac{(8\sqrt{2})(-4+2\sqrt{2})}{(-4-2\sqrt{2})(-4+2\sqrt{2})} = \frac{(8\sqrt{2})(-4+2\sqrt{2})}{8} = 4-4\sqrt{2}$, so the length is $4\sqrt{2}-4$ cm.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 5 Round 4
Quadratic Equations and Complex Numbers

1.) ______ $\pm 3i, \pm 2i$ _________

2.) _______ $\frac{1}{3z+1}, -1$ ______

3.) ___ $2.5-3i, -2.5+3i$ _____

1.) If $y^2 = x$, find all complex values of y for which $x^2+13x+36=0$.
This factors to (x+9)(x+4)=0,
So $y^2=-9$, and $y^2=-4$, so y= $\pm 3i, \pm 2i$

2.) Solve for w in terms of z if $z \neq \frac{-1}{3}$: $(3z+1)w^2+(3z)w+2=3$

This becomes $(3z+1)w^2+(3z)w-1=0$.

$$w=\frac{-3z\pm\sqrt{(3z)^2-4(-1)(3z+1)}}{6z+2}\quad w=\frac{-3z\pm\sqrt{9z^2+12z+4}}{6z+2}$$

$$w=\frac{-3z\pm|3z+2|}{6z+2}, w=\frac{-3z\pm(3z+2)}{6z+2}=$$

$$\frac{-3z+(3z+2)}{6z+2}, \frac{-3z+(-3z-2)}{6z+2}$$

w= $\frac{1}{3z+1}, -1$

3) Find the two complex square roots of $\frac{-11}{4}-15i$

$$\sqrt{\frac{-11}{4}-15i}=a+bi, \frac{-11}{4}-15i=(a^2-b^2)+2abi,$$

$$a^2-b^2=\frac{-11}{4}, 2ab=-15.$$

$$b=\frac{-15}{2a}, a^2-(\frac{-15}{2a})^2=\frac{-11}{4}, 4a^4-225=-11a^2$$

$$4a^4+11a^2-225=0, (4a^2-25)(a^2+9)=0,$$

so a=±2.5. If a=2.5, b=-3, and if a=-2.5, b=3. Solutions are 2.5-3i, -2.5+3i

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 5 Round 5
Solving Trig Equations

1.) ________ $0, \frac{2p}{3}, \frac{4p}{3}$ ________

2.) _______ $\frac{5p}{6}, \frac{p}{6}, \frac{p}{2}$ ________

3.) ____ $\frac{1 \pm 6\sqrt{2}}{10}$ ______

1) Solve for all x if $0 \leq x < 2\pi$: $2\cos^2(x) - \cos(x) = 1$
   $2\cos^2(x) - \cos(x) - 1 = 0$, so $(2\cos x + 1)(\cos x - 1) = 0$, so
   $\cos x = -1/2$ or $\cos x = 1$, so x = $0, \frac{2p}{3}, \frac{4p}{3}$

2) Solve for x if $0 < x < \pi$: $\csc(x)\cot^2(x) = \csc^2(x) + \frac{3}{\sin(x)} - 4$

   $\csc(x)(\csc^2(x)-1) = \csc^2(x) + 3\csc(x) - 4$
   $\csc^3(x) - \csc(x) = \csc^2(x) + 3\csc(x) - 4$
   $\csc^3(x) - \csc^2(x) - 4\csc(x) + 4 = 0$
   $\csc^2(x)(\csc(x)-1) - 4(\csc(x)-1) = 0$
   $(\csc^2(x)-4)(\csc(x)-1) = 0$
   $(\csc(x)+2)(\csc(x)-2)(\csc(x)-1) = 0$
   $\csc(x) = -2$ or $\csc(x) = 2$ or $\csc(x) = 1$
   $\sin x = -1/2$ or $\sin x = ½$ or $\sin x = 1$
   x= $\frac{5p}{6}, \frac{p}{6}, \frac{p}{2}$

3) Find all values of sin(A) such that $\sin(A + \frac{p}{3}) = \frac{1}{5}$

$$\sin(A + \frac{p}{3}) = \sin A * \cos(\frac{p}{3}) + \cos A * \sin(\frac{p}{3})$$

$$= \frac{\sin A}{2} + \frac{\sqrt{3}\cos A}{2}$$

Solve $\frac{\sin A}{2} + \frac{\sqrt{3}\cos A}{2} = \frac{1}{5}$, so $\frac{\sqrt{3}\cos A}{2} = \frac{1}{5} - \frac{\sin A}{2}$. Square both sides to get

$\frac{3\cos^2 A}{4} = \frac{1}{25} - \frac{\sin A}{5} + \frac{\sin^2 A}{4}$, so $\frac{3(1 - \sin^2 A)}{4} = \frac{1}{25} - \frac{\sin A}{5} + \frac{\sin^2 A}{4}$, so

$\sin^2 A - \frac{\sin A}{5} - \frac{71}{100} = 0$, so $100\sin^2 A - 20\sin A - 71 = 0$, so

$$\sin A = \frac{20 \pm \sqrt{400 + 28400}}{200} = \frac{20 \pm \sqrt{28800}}{200} = \frac{20 \pm 120\sqrt{2}}{200} = \frac{1 \pm 6\sqrt{2}}{10}$$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 5 Round 6
Sequences and Series

1.) ______196__________

2.) ________ $\frac{-109}{9}$, 3_____________

3.) ________256, 4_______________

1.) The sequence $\{L_n\}$ is defined recursively as follows: $L_1=1$, $L_2=3$, and for $n>2$, $L_n=L_{n-1}+L_{n-2}$. Evaluate $\sum_{n=1}^{9} L_n$.

This is the Lucas sequence. The first nine terms are 1,3,4,7,11,18,29,47,76. Their sum is 196.

2.) In an arithmetic sequence, the ninth term is 6 less than the square of the second term. If the fifth term is 11, find all possible values for the first term of the sequence.

$a_9 = a_1 + 8d = (a_1 + d)^2 - 6.$
$a_5 = 11 = a_1 + 4d \rightarrow a_1 = 11 - 4d$ Substitute to get

, $(11-4d)+8d=(11-4d+d)^2-6$. so $11+4d=(11-3d)^2-6$, so $11+4d=121-66d+9d^2-6$, so $9d^2-70d+104=0$, so $(d-2)(9d-52)=0$, so $d=2$ or $d=\frac{52}{9}$, so 11-2*4=3 or 11-(52/9)*4= (99-208)/4 = -109/9

3.) For a geometric sequence $\{a_n\}$ of real numbers, $\sum_{n=1}^{\infty} a_n = 3125.$

If $a_2 = 500$, what are all possible values for the fifth term of the sequence?

$a_1 r = 500$ and $\frac{a_1}{1-r} = 3125$, so $\frac{500}{(1-r)r} = 3125$ and

$(1-r)r = \frac{500}{3125} = 0.16$, so $r^2-r+0.16=0$, $(r-0.2)(r-0.8)=0$, so $r=0.2$ or $r=0.8$.

If $r=0.2$, $a_5=500(0.2)$^3= 4. If $r=0.8$, $a_5=500(0.8)$^3=256.

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 5
Team
Round

1.) _(4,2), (8,2), (8,4), (9,3)__

2.) _______ $\frac{7 \pm \sqrt{85}}{6}$ _______

3.) __________ 9 _______

4.) ____ $-35-5i$ __________

5.) ___ $\frac{-4 \pm \sqrt{7}}{3}$ ________

6.) _______ $\frac{-1}{12}, \frac{16}{3}$ _______

1) If a and b are integers such that 2≤a≤10 and 2≤b≤10, find all ordered pairs (a,b) such that the following expression is an integer: Give your answers as ordered pairs (a,b).

$$\frac{(77a^{2014})(13b^{-6})}{(a^{503})^4 b^{-3}} - \frac{(10a^{15})^3 (b^{10})^6}{a^{43}(b^9)^7}$$

$$\frac{(77a^{2014})(13b^{-6})}{(a^{503})^4 b^{-3}} - \frac{(10a^{15})^3 (b^{10})^6}{a^{43}(b^9)^7} =$$

$$\frac{1001a^{2014}b^{-6}}{a^{2012}b^{-3}} - \frac{1000a^{45}b^{60}}{a^{43}b^{63}} = \frac{a^2}{b^3}$$

.

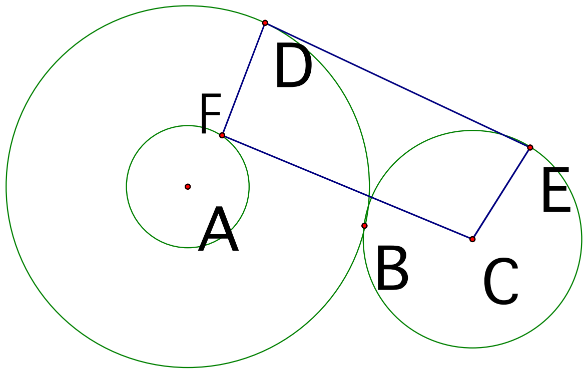
The only combinations within the given range are a=4, b=2; a=8, b=2; a=8, b=4; a=9, b=3

2.) Solve:

$$2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{x}}} = x, 2 + \frac{1}{1 + \frac{1}{\frac{2x+1}{x}}} = x, 2 + \frac{1}{1 + \frac{x}{2x+1}} = x$$

$$2 + \frac{1}{\frac{3x+1}{2x+1}} = x, 2 + \frac{2x+1}{3x+1} = x, \frac{8x+3}{3x+1} = x,$$

so x(3x+1)=8x+3, so $3x^2$+x=8x+3, so $3x^2$-7x-3=0, so x= $\frac{7 \pm \sqrt{85}}{6}$



3.) A circle with center A has radius x cm. A circle with center C has radius 5 cm. The two circles are externally tangent to one another at B. Circle A is tangent to line DE at D and circle C is tangent to line DE at E. The length of DE is $6\sqrt{5}$ cm. Find the radius of the circle with center A.

Solution: Solution: Draw a circle of radius x-5 with center A. Suppose AD meets the new circle at F. The length of FC is the same as the length of DE and since F lies on AD, ΔAFC is a right triangle. $(AF)^2+(AC)^2 = (FC)^2$, and so $(x-5)^2 + (6\sqrt{5})^2 = (x+5)^2$, and $X^2-10x+25+180=x^2+10x+25$, so 20x=180, and x=9.

4.) Express in a+bi form for a, b real numbers:

$$\frac{125}{(1+2i)^3}+\frac{625i}{(2-i)^4}=$$

$$\frac{125}{(-3+4i)(1+2i)}+\frac{625i}{(3-4i)(3-4i)}=$$

$$\frac{125}{-11-2i}+\frac{625i}{-7-24i}=$$

$$\frac{125(-11+2i)}{(-11-2i)(-11+2i)}+\frac{625i(-7+24i)}{(-7-24i)(-7+24i)}=$$

$$\frac{125(-11+2i)}{125}+\frac{625i(-7+24i)}{625}=-11+2i+(-7i-24)=-35-5i$$

5) Find all possible values of tan(x) such that sec(x) – 2 tan(x) = 2.

Sec(x) = 2 tan(x) + 2. Square both sides to get $\sec^2(x) = 4\tan^2(x) + 8\tan(x) + 4$.
Substitute $\sec^2(x) = \tan^2(x)+1$ to get $\tan^2(x)+1= 4\tan^2(x) + 8\tan(x) + 4$, so
$3\tan^2(x) + 8\tan(x) + 3=0$, so $\tan(x) = \frac{-8\pm\sqrt{64-36}}{6} = \frac{-8\pm\sqrt{28}}{6} = \frac{-8\pm2\sqrt{7}}{6} = \frac{-4\pm\sqrt{7}}{3}$

6.) In a geometric sequence of complex numbers, the 5th term is 4 less than twice the third term. If the 7th term is 18, what are all possible values for the first term of the sequence?

$t_5 = t_3 r^2 = 2t_3 - 4$ and $t_7 = 18 = t_3 r^4$, so $r^2 = \pm\sqrt{\frac{18}{t_3}}$, but we'll square both sides to find t_3 anyway, so $t_3\sqrt{\frac{18}{t_3}} = 2t_3 - 4$, so $\sqrt{18t_3} = 2t_3 - 4$, so , $18t_3 = 4t_3^2 - 16t_3 + 16$

$4t_3^2 - 34t_3 + 16 = 0$, so , $2t_3^2 - 17t_3 + 8 = 0$, $(2t_3 - 1)(t_3 - 8) = 0$, $t_3=1/2$ or $t_3=8$.

If the third term is 8, we have $8r^2=12$, so $t_1 = \frac{8}{1.5} = \frac{16}{3}$.

If the third term is ½, we have $(1/2)r^2=-3$, so $r^2=-6$, so $t_1 = \frac{\frac{1}{2}}{-6} = \frac{-1}{12}$