FAIRFIELD COUNTY MATH LEAGUE (FCML) 2012-2013

Match 4 Round 1
Arithmetic:
Basic Statistics

1.) __24.8 (or_24 4/5 or 124/5)

2.) ______40__________

3.) _______0______________

1 point: The geometric mean of the numbers $x_1, x_2, \ldots x_n$ is $\sqrt[n]{x_1 x_2 \ldots x_n}$. What is the product of the arithmetic mean and the geometric mean of the set of numbers {2, 8, 1, 16, 4} ?

Solution: The geometric mean is the 5th root of 2*8*1*16*4 = the fifth root of 2^10, which is 2^2 =4. The arithmetic mean is (2+8+1+16+4)/5 = 31/5 = 6.2. 6.2x4 = 24.8

2 points: The interquartile range of a set of numbers is the positive difference between its upper quartile and its lower quartile. A set of 55 numbers consists of 1 10, 2 20's, 3 30's, 4 40's, and so on up to 10 100's. What is the interquartile range of this set of numbers? (Answer: 40)

Solution: The median is the 28th number, the lower quartile is the 14th number, and the upper quartile is the 42nd number, when the numbers are put in order. The 14th number must be 50, since the set goes 10,20,20,30,30,30,40,40,40,40,50,50,50,50,50,…
And the 42nd number must be 90, since working backwards, numbers 46 through 55 are all 100, and numbers 37 through 45 are 90. So 90-50 = 40

3 points: A set of 10 distinct numbers which have the same mean and median is arranged in ascending order. Any two numbers in the set differ by at least 10. 1 is added to the smallest number, 2 is added to the second number, 3 is added to the third number, and so on, through 10 added to the largest number. What is the difference between the new mean and the new median

Solution: Since all the numbers differ by at least 10, adding a number between 1 and 10 won't affect the order of the numbers. The new median will be 5.5 more than the old median, since 5 was added to the 5th number and 6 was added to the 6th number. However, the new mean differs from the old mean by 5.5 also, since this is (1+2+3+4+5+6+7+8+9+10)/10, so the difference is 0.

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Match 4 Round 2
Algebra 1: Quadratic Equations

1.) ____7/11 or 13/5_____

2.) _______-192______________

3.) ____$4x^2 - 73x + 144 = 0$

1 point: Find the two solutions of the equation $55x^2 - 178x + 91 = 0$ (Answer: 7/11 and 13/5)

Solution: This factors to $(11x-7)(5x-13) = 0$, so $x=7/11$ or $13/5$

2 points: Find the value of k such that the equation $4(x+2)^2 - 3(x-2)^2 = k$ has exactly one solution.

Solution: This is $4x^2 + 16x + 16 - (3x^2 - 12x + 12) = k$, so $x^2 + 28x + 4 = k$, so $x^2 + 28x + (4-k) = 0$. The discriminant b^2-4ac must be zero, so $28^2 - 4(4-k)=0$, so $784-16+4k = 0$, so $4k = -768$, so $k=-192$.

3 points: Find a quadratic equation whose solutions are the reciprocals of the solutions of the equation $144x^2 - 73x + 4 = 0$. Express your equation as $ax^2 + bx + c = 0$ with $a>0$, and a,b,c are relatively prime.

Solution: $144x^2 - 73x + 4$ factors to $(16x-1)(9x-4)$ so the solutions are 1/16 and 4/9. We need an equation whose solutions are 16 and 9/4. The sum of the two solutions must be 73/4. The product of the solutions is 144/4, so the desired equation is $4x^2 - 73x + 144 = 0$.

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Match 4 Round 3
Geometry:
Similarity

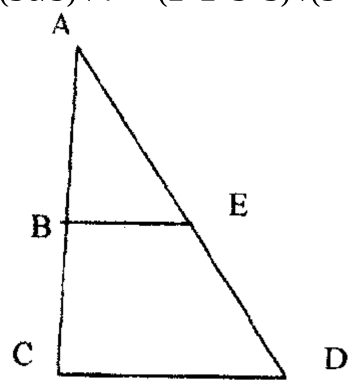
1.) $\frac{81}{25}$

2.) $\frac{145}{6}$

3.) $\frac{12}{35}$

1 point: Two regular pentagons are such that the larger pentagon has perimeter 36 and the length of one side of the smaller pentagon is 4. What is the ratio of the area of the larger pentagon to the area of the smaller pentagon (give two relatively prime integers)

Solution: Each side of the larger pentagon has length 36/5 = 7.2. The ratio of the areas is $(36/5)^2/4^2 = (2*2*3*3)^2/(5*5*2*2*2*2) = (3*3*3*3)/(5*5) = 81/25$



2 points: In the figure above, $\overline{BE} \parallel \overline{CD}$. If BC =4, ED=6, AD=10, and, BE = 3, find the perimeter of ΔACD.

Solution: AE = AD-ED, so AE = 4. AB/AC = AE/AD = BE/CD, so AB/(4+AB) = 4/10, so 16+4(AB) = 10(AB), so AB= 16/6 = 8/3. Then AC = AB+BC=8/3+4 = 20/3. Using AB/AC = BE/CD, we have (8/3)/(20/3) = 3/CD, so 8/20=3/CD, so 8(CD)=60, so CD = 60/8 = 15/2. Adding AC + CD + AD, we get 20/3 + 15/2 + 10 = 40/6 + 45/6 + 60/6 = 145/6

3 points: Right triangle ABC has sides 3,4, and 5 with the AB as the hypotenuse. The altitude drawn from C intersects AB at D. The bisector of angle ACB intersects AB at E. Find the length of DE. (Answer: 12/35)

Solution: The altitude will make all the triangles similar, so it will need to split D into a ratio 1.8/3.2 so that the length of the altitude is 2.4, and you have the 2 smaller triangles as 1.8-2.4-3 and 2.4-3.2-4. The angle bisector splits AB into the same ratio as the sides, so 5 has to be split in a 3-4 ratio, so it must be 15/7 and 20/7. The distance between D and E must be 15/7-9/5 = 75/35-63/35 = 12/35

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Match 4 Round 3
Algebra 2:
Variation

1.) ___________9___________

2.) _________$240\sqrt{6}$_______________

3.) _______$20\sqrt{5}$_________________

1.) The volume V of a gas varies directly with its temperature T in Kelvins and inversely with its pressure P. A certain gas occupies 10 liters of volume at standard temperature and pressure (Standard temperature and pressure are considered 273 Kelvins and 1 atmosphere). To the nearest liter, what is the volume of the gas at 300 Kelvins and 1.2 atmospheres.

Solution: V=kT/P, so PV/T is constant. 10*1/273 = V*1.2/300.
So 300*10/(273*1.2) = V. V = 3000/327.6, which is 9 liters to the nearest liter.

2 points: x varies directly with y, and y varies inversely with z^3. When x=5, z=$4\sqrt{3}$.
What is x when z = $\sqrt{2}$?

Solution: Since x varies directly with y, x also varies inversely with z^3. $5 = k/(4\sqrt{3})^3$,
So $k = 5*4^3*3*\sqrt{3} = 960\sqrt{3}$. Then $x = (960\sqrt{3})/(\sqrt{2})^3 = (480\sqrt{3})/(\sqrt{2}) = 240\sqrt{6}$

3 points: The energy in joules stored in a electrical capacitor varies jointly with its capacitance in farads and the square of the potential difference between the two plates of the capacitor in volts. If a capacitor of $3x10^{-6}$ farads has a 4 volt difference between the plates, the energy stored is 2.4×10^{-5} joules. If the same capacitor stores 0.003 joules of energy, what is the potential difference between the plates in volts?

Let E=energy, C=capacitance, V=potential, so $E=kCV^2$. $2.4x10^{-5} = k(3x10^{-6})(4^2)$, so
$k=\frac{2.4x10^{-5}}{4.8x10^{-5}} = 0.5$, so $E = 0.5CV^2$. Then $0.003 = 0.5(3x10^{-6})V^2$, so $V^2 =$
$\frac{0.003}{0.5*3x10^{-6}} = \frac{0.003}{1.5x10^{-6}} = 2000$. Then $V = \sqrt{2000} = \sqrt{400}\sqrt{5} = 20\sqrt{5}$ volts.

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Match 4 Round 5
Trig Expressions

1.) _____$\sin^2(x)$_____

2.) _$-8\sin^2(x) + 6\sin(x)$______

3.) ________ $\dfrac{3\sqrt{3}+88}{9}$ __________

1.) Express in terms of a single trig function of x with no functions in the denominator:

$$\sin\left(\frac{\pi}{2}-x\right)\cos(\pi-x)\cot\left(\frac{3\pi}{2}-x\right)\tan(2\pi-x)$$

Solution: This is cos(x)*-cos(x)*tan(x)*(-tan(x)) = cos(x)*-cos(x)*$\dfrac{\sin(x)*(-\sin(x))}{\cos(x)*\cos(x)}$

= $\sin^2(x)$

2. Simplify and express in terms of sin(x): $\dfrac{\sec(3x)}{\csc(6x)}$

$$\frac{\sec(3x)}{\csc(6x)} = \frac{\sin(6x)}{\cos(3x)} = \frac{2\sin(3x)\cos(3x)}{\cos(3x)} =$$

$$2\sin(3x) = 2\sin(2x+x) = 2(\sin(2x)\cos(x)+\cos(2x)\sin(x)) =$$

$$2\left[(2\sin x\cos x)(\cos x)+\left(2\cos^2 x-1\right)\sin x\right] = 8\cos^2 x\sin x - 2\sin x$$

$$= 8(1-\sin^2 x)(\sin x) - 2\sin x = -8\sin^2 x + 6\sin x$$

2.) If x = $\dfrac{\pi}{3}$, evaluate $\cot(x) + \tan^2(3x) + \sec^3(5x) + \csc^4(7x)$

Solution:

$$\frac{\sqrt{3}}{3}+0-(-2)^3+\left(\frac{2\sqrt{3}}{3}\right)^4 = \frac{\sqrt{3}}{3}+0+8+\left(\frac{144}{81}\right)$$

$$= \frac{\sqrt{3}}{3}+0+8+\left(\frac{16}{9}\right) = \frac{3\sqrt{3}}{9}+0+\left(\frac{72}{9}\right)+\left(\frac{16}{9}\right) = \frac{3\sqrt{3}+88}{9}$$

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Match 4 Round 5
Conics

1.) ______ $\left(1, -0.5 - \sqrt{5}\right)$ __________

2.) __________ $\frac{1}{2}$ ______________

3.) ____________ $16\sqrt{3}$ __________

1) Find the location of the focus of the ellipse $9x^2 + 4y^2 – 18x + 4y – 26 = 0$ that is farthest from the origin.
This is $9x^2 – 18x + 9 + 4y^2 + 4y + 1 = 36$, so
$\frac{(x-1)^2}{4} + \frac{(y+0.5)^2}{9} = 36$, so the center of the ellipse is at (1,-0.5) and the foci are $\sqrt{(9-4)}$ units from the center in the vertical direction, so the one that is the farthest distance from the origin is at $(1, -0.5-\sqrt{5})$

2) The focus of the parabola $y=bx^2$ is 8 times farther from the origin than the focus of the parabola $x=ay^2$. If the two parabolas intersect at a point whose y-coordinate is $\frac{1}{4}$, what is the x-coordinate of the point of intersection?
Since the focus of $y=bx^2$ is 8 times farther from the origin than $x=ay^2$, then (1/(4b))=8(1/(4a)), so a=8b. ¼ is a common solution, so $x = (8b)(1/4)^2$ or (1/2)b and $(1/4) = (b)x^2$, so $(1/4) = (b)[(1/2)b]^2$, so $b^3=1$, and then b=1. Since $y=x^2$, solve $(1/4)=x^2$ to get x = ½

3) A hyperbola with its foci at $\left(\sqrt{5}, 0\right) and \left(-\sqrt{5}, 0\right)$ passes through four points of the circle $x^2 + y^2 = 16$. One of the asymptotes of the hyperbola has slope 2. What is the area of the rectangle formed by connecting the four points of intersection?
Since an asymptote has slope 2 and the foci are such that the center of the hyperbola is the origin, the equation of the hyperbola must be $\frac{x^2}{a^2} - \frac{y^2}{4a^2} = 1$ for some value of a. Since the foci are $\sqrt{5}$ units away from the center, $a^2 + (4a^2) = 5$, so a=1, so the equation of the hyperbola is $\frac{x^2}{1} - \frac{y^2}{4} = 1$, or $4x^2 – y^2 = 4$. Solve simultaneously with $x^2 + y^2 = 16$ so get $5x^2 = 20$, so $x = \pm 2$, and then $y = \pm 2\sqrt{3}$. The rectangle has sides 4 and $4\sqrt{3}$, so they multiply to $16\sqrt{3}$.

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Match 4 Team Round

1.) ______ $\frac{5-5\sqrt{3}}{12}$ _____

2.) ___ $\frac{p}{3}, p, \frac{5p}{3}$ _________

3.) ________12_______________

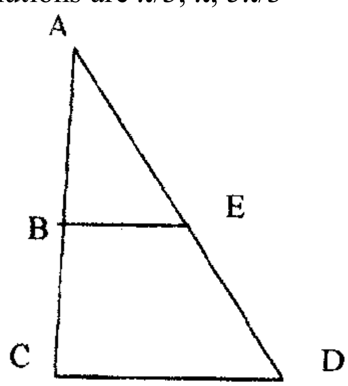
4.) __________ $\frac{18\sqrt{5}-30}{5}$ ________

5.) _____ $y = \frac{\sqrt{3}}{(\sin x)^3}$ ______

6.) _____________ $\frac{2}{5}$ __________

1.) What is the arithmetic mean of the 6 numbers sin(5π/6), cos(5π/6), tan(5π/6), cot(5π/6), sec(5π/6), and csc(5π/6)?
(1/2 -√3/2 - √3/3 - √3 -(2√3)/3 + 2)/6 = (3/6 - (3√3)/6 – (2√3)/6 – (6√3)/6 - (4√3)/6 + 12/6)/6 = (15-15√3)/36 = (5 – 5√3)/12

2) Find all the solutions of the equation $\tan^2 x$ - sec x - 1 = 0 in the interval [0,2π)
$\tan^2 x + 1 = \sec^2 x$, so this becomes ($\sec^2 x$-1) – sec x – 1 = 0, so $\sec^2 x$ – sec x – 2 = 0, so (sec x -2)(sec x + 1) = 0, so sec x = 2 or sec x = -1, so cos x = ½ or cos x = -1, so the solutions are π/3, π, 5π/3



3) In the figure above, BE || CD. BE is 2 more than $\frac{1}{2}$ times CD. AB is 15 less than 3 times BC. CD is twice as long as BC. Find the length of BE.
Let x = BC. Then 2x = CD. We have $\frac{AB}{AC} = \frac{BE}{CD}$, so $\frac{AB}{AB+BC} = \frac{BE}{CD}$, so

$\frac{3x-15}{3x-15+x} = \frac{2+0.5(2x)}{2x}$ $\frac{3x-15}{4x-15} = \frac{2+x}{2x}$ $\frac{3x-15}{3x-15+x} = \frac{2+0.5(2x)}{2x}$, so

$\frac{3x-15}{4x-15} = \frac{2+x}{2x}$, so $6x^2 – 30x = 4x^2 -7x – 30$, so
$2x^2 – 23x – 30 = 0$, so (2x-3)(x-10) = 0. Reject x=3/2 because that would make the length of AB negative, so x=10. Then CD = 20, AB=15, and BE = 12.

4) The asymptotes of a hyperbola are tangent to the circle $x^2 + y^2 = 9$ at the points $(2, \sqrt{5})$ and $(-2, \sqrt{5})$. If one of the y-intercepts of the hyperbola is on the circle, what is the smallest possible distance between the two y-intercepts of the hyperbola?

Since the asymptotes are tangent to the circle at the given points, they must slopes that are the negative reciprocals of the lines connecting the given points to the origin (√5/2 and -√5/2), which gives (-2/√5) and (2/√5). The center of the hyperbola must be located on the y-axis, since its asymptotes are tangent to two points that are equidistant from the y-axis. Find the center of the hyperbola by letting x=0 in y-√5 = (-2/√5)(x-2), so y=√5+4/√5 = (5√5+4√5)/5 = (9√5)/5. The circle intersects the y-axis at (0,3) and (0,-3), so the smallest distance between the intercepts is when y=3. In that case the distance is twice the difference between the y-coordinate of the center of the hyperbola and 3, so

2(9√(5)/5 – 3) = 2(9√5-15)/5 = $\dfrac{18\sqrt{5} - 30}{5}$

5) y=f(x) is defined for $0 \le x \le \dfrac{p}{2}$. y varies inversely with some power of sin(x). If $\left(\dfrac{p}{6}, 8\sqrt{3}\right) and \left(\dfrac{p}{3}, \dfrac{8}{3}\right)$ both belong to the function f(x), express the function as $y = \dfrac{k}{(\sin x)^n}$ with numbers in place of k and n.

If x=π/6, we have 8√3 = a/((1/2)^n, and if x=π/3 we have 8/3 = a/(√3/2)^n.
Divide the first equation by the second to get (√3)^n = 3√3, so n=3. Substituting into the first equation, 8√3 = a/(1/2)^3 , so 8√3 = 8a, so a=√3. $y = \dfrac{\sqrt{3}}{(\sin x)^3}$ $y = \dfrac{\sqrt{3}}{(\sin x)^3}$

6) y varies directly with the square of x+1. For some value x=k, y=2. If x=k+5, then y=8. What is the median of all possible values for the constant of variation?
Let a be the constant of variation. Then
2=a(k+1)^2, and 8 = a(k+5+1)^2, or a(k+6)^2. Divide the second equation by the first to get 4 = (k+6)^2/(k+1)^2, so 4(k+1)^2 = (k+6)^2. Solve for k to get 4k^2 + 8k + 4 = k^2 + 12k + 36, so 3k^2 – 4k – 32 = 0, so (3k+8)(k-4)=0, so k= -8/3 or k=4. If k=-8/3, then 2 = a(-5/3)^2, so a = 2(9/25) = 18/25. If k=4, then 2 = a(5)^2, so a = 2/25. The median of these is 10/25 = 2/5.