FAIRFIELD COUNTY MATH LEAGUE 2018-2019

Match 1 Round 1
Arithmetic: Percents

1.) $x = 15$

2.) $2000\sqrt{6}$

3.) 86.4

1.) If increasing x by 20 percent has the same result as decreasing the quantity 5 more than x by 10 percent, find the value of x.

$1.2x = .9(x + 5)$

$1.2x = .9x + 4.5$

$.3x = 4.5$

$x = \frac{4.5}{.3} = 15$

2.) A rectangular prism has dimensions a, b, and c. If a percent of b is 3, b percent of c is 4, and c percent of a is 2, find the exact volume of the prism.

We know that $\frac{a}{100}b = 3$, $\frac{b}{100}c = 4$, and $\frac{c}{100}a = 2$, which means $ab = 300$, $bc = 400$, and $ac = 200$. The volume of the prism is abc, and $abc = \sqrt{a^2b^2c^2} = \sqrt{(ab)(bc)(ac)} = \sqrt{(300)(400)(200)} = \sqrt{24 * 10^6} = 10^3(2\sqrt{6}) = 2000\sqrt{6}$.

3.) Mr. Purse and Ms. Cents each give a test to one of their classes (which do not have the same number of students). The average (arithmetic mean) test score in Mr. Purse's class is 80 and the average test score in Ms. Cents's class is 90. Ms. Cents notes that if the average score in Mr. Purse's class were to decrease by 20% and the average score in Ms. Cents's class were to increase by 10%, the combined average would not change. State the exact combined average of both classes as a decimal.

We will let p represent the number of students in Mr. Purse's class and c represent the number of students in Ms. Cents's class. Based on the first statement, we know that $\frac{.8(80p)+1.1(90c)}{p+c} = \frac{80p+90c}{p+c}$, which gives $64p + 99c = 80p + 90c \rightarrow 9c = 16p \rightarrow c = \frac{16}{9}p$. Plugging this back in to find the average:

$$\frac{80p + 90(\frac{16}{9}p)}{p + \frac{16}{9}p} = \frac{80p + 160p}{\frac{25}{9}p} = \frac{240}{\frac{25}{9}} = \frac{9(240)}{25} = 86.4$$

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Match 1 Round 2
Algebra 1: Equations

1.) $x = 5$

2.) $y = 4$

3.) $x = -\frac{25}{13}$

1.) Find the value of x: $\frac{2}{7}(3x - 1) = 4\big(x - 2(x - 3)\big)$

$$\frac{6}{7}x - \frac{2}{7} = 4(x - 2x + 6)$$
$$\frac{6}{7}x - \frac{2}{7} = -4x + 24$$
$$\frac{34}{7}x = \frac{170}{7}$$
$$x = \frac{170}{34} = 5$$

2.) Find the value of y: $\frac{y-2}{y-1} = \frac{y}{y+2}$

$$(y - 2)(y + 2) = y(y - 1)$$
$$y^2 - 4 = y^2 - y$$
$$-4 = -y$$
$$y = 4$$

3.) If $(x + 2)(x + 4)(x - 3) = (x + 1)^3$, find the value of x.

$$x^3 + 3x^2 - 10x - 24 = x^3 + 3x^2 + 3x + 1$$
$$-10x - 24 = 3x + 1$$
$$-13x = 25$$
$$x = -\frac{25}{13}$$

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Match 1 Round 3 Geometry: Triangles & Quadrilaterals

1.) $8\sqrt{10}$

2.) $\frac{\sqrt{15}}{5}$

3.) $\frac{60}{7}$

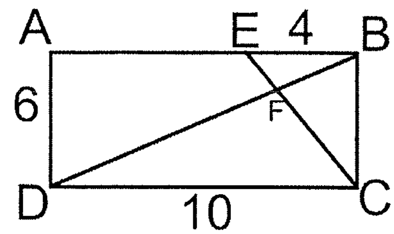
1.) Find the perimeter of a rhombus with area 24 and one diagonal of length 12.

We can solve for the length of the second diagonal by noting that $24 = \frac{1}{2}d_1d_2 \rightarrow 24 = \frac{1}{2}(12)d_2 \rightarrow d_2 = 4$. This makes each side of the rhombus the hypotenuse of a right triangle with legs of lengths 2 and 6. Since $2^2 + 6^2 = 40$, each side has length $2\sqrt{10}$, making the perimeter $8\sqrt{10}$.

2.) A rectangle whose length is three times its width has the same area as a given square. Find the ratio of the length of one diagonal of the square to the length of one diagonal of the rectangle. Give your answer as a simplified rational expression.

If we assume that the rectangle has sides of length x and $3x$ and therefore an area of $3x^2$, the square would have sides of length $x\sqrt{3}$. We can use the Pythagorean theorem to find that the length of one diagonal of the square has length $x\sqrt{6}$ and one diagonal of the rectangle has length $x\sqrt{10}$. The ratio is therefore $\frac{x\sqrt{6}}{x\sqrt{10}} = \frac{\sqrt{15}}{5}$.

3.) Consider rectangle *ABCD* with point *E* on $\overline{AB}$. $\overline{EC}$ intersects diagonal $\overline{BD}$ at point *F*. If $AD = 6$, the area of rectangle *ABCD* is 60, and the area of trapezoid *AECD* is 48, find the exact area of triangle *BFC*.



We are given that $AD = 6$ and the area of the rectangle is 60, so we know $AB = CD = 10$. We can use the area of the trapezoid ($48 = \frac{1}{2}(6)(10 + b)$) to find that $AE = 6$ and therefore $EB = 4$. Next we can note that $\Delta EBF \sim \Delta CDF$ and that $CD = \frac{5}{2}EB$. We can then solve for the altitude of triangle *EBF* perpendicular to $\overline{EB}$ using $x + \frac{5}{2}x = 6$, giving $x = \frac{12}{7}$. We then know the area of triangle *EBF* is $\frac{1}{2}(4)\left(\frac{12}{7}\right) = \frac{24}{7}$. Finally we can solve for the area of triangle *BFC* by subtracting this area from 12, the area of triangle *EBC*: $12 - \frac{24}{7} = \frac{60}{7}$.

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Match 1 Round 4
Algebra 2: Simultaneous Equations

1.) $y = 5$

2.) $\left(-\frac{13}{5}, \sqrt{2}\right)$ $\left(-\frac{13}{5}, -\sqrt{2}\right)$

3.) $\left(\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{2}{3}, -\frac{2}{5}\right)$

1.) Solve the following system for y: $\begin{cases} 2x + 3y = 11 \\ y = x + 7 \end{cases}$

Substituting $x = y - 7$ in the first equation yields $2(y-7) + 3y = 11$. Solving for y gives $y = 5$.

2.) Find all solutions as ordered pairs (x, y): $\begin{cases} \frac{2}{x+3} - y^2 = 3 \\ \frac{4}{x+3} + 3y^2 = 16 \end{cases}$

Adding 3 times the first equation to the second equation eliminates the y variable, leaving $\frac{10}{x+3} = 25$. Solving for x gives $x = -\frac{13}{5}$. Substituting back into either equation and solving for y^2 yields $y = \pm\sqrt{2}$. This gives the 2 ordered pair solutions of $\left(-\frac{13}{5}, \sqrt{2}\right)$ and $\left(-\frac{13}{5}, -\sqrt{2}\right)$.

3.) Find all solutions as ordered pairs (a, b): $\begin{cases} 3a - 5b = 4 \\ \frac{1}{a} - \frac{1}{b} = 4 \end{cases}$

Note that $3a = 5b + 4$. The second equation can be rewritten as $\frac{3}{3a} - \frac{1}{b} = 4$.

Substitution yields $\frac{3}{5b+4} - \frac{1}{b} = 4$. Multiplication produces the quadratic $3b - (5b + 4) = 4b(5b + 4) \rightarrow 10b^2 + 9b + 2 = 0 \rightarrow$ $(2b + 1)(5b + 2) = 0$. If $b = -\frac{1}{2}$, we can solve for a using either equation and we get $a = \frac{1}{2}$. If $b = -\frac{2}{5}$, we get $a = \frac{2}{3}$.

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Match 1 Round 5
Precalculus: Right Triangle Trigonometry

1.) $2\sqrt{2}$

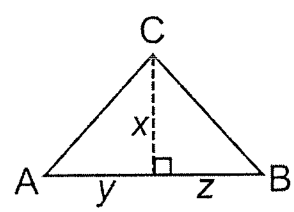
2.) $\frac{27}{14}$

3.) $\frac{5\sqrt{33}}{32}$

1.) If m is an acute angle measure in degrees and $\sin(m) = \frac{1}{3}$, find $\tan(90 - m)$.

Note that $\cos(m) = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$. Given $\tan(90 - m) = \cot(m)$, we know $\tan(90 - m) = \frac{\cos(m)}{\sin(m)} = 2\sqrt{2}$.

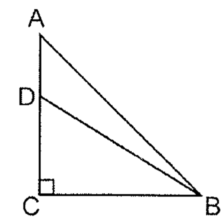
2.) Consider acute triangle ABC with an area of 21. If $AB = 9$, find the value of $\cot(A) + \cot(B)$.



Consider the diagram to the right. From the diagram, $\cot(A) = \frac{y}{x}$ and $\cot(B) = \frac{z}{x}$, so $\cot(A) + \cot(B) = \frac{y+z}{x}$. It is given that $y + z = AB = 9$, and $\frac{1}{2}(9)x = 21 \rightarrow x = \frac{14}{3}$. Therefore $\frac{y+z}{x} = \frac{9}{\frac{14}{3}} = \frac{27}{14}$.

3.) Consider triangle ABC with right angle C and point D on $\overline{AC}$. If $\tan(\angle BAC) = \frac{5}{8}$ and $\sin(\angle BDC) = \frac{4}{7}$, find $\frac{\text{area } \Delta BDC}{\text{area } \Delta ABC}$.

Consider the diagram to the right. Since $\frac{BC}{AC} = \frac{5}{8}$ and $\frac{BC}{BD} = \frac{4}{7}$, we can arbitrarily set $BC = 20$, making $AC = 32$ and $BD = 35$. Using Pythagorean Theorem yields $CD = 5\sqrt{33}$. This means $\frac{\text{area } \Delta BDC}{\text{area } \Delta ABC} = \frac{\frac{1}{2}(20)(5\sqrt{33})}{\frac{1}{2}(20)(32)} = \frac{5\sqrt{33}}{32}$.



FAIRFIELD COUNTY MATH LEAGUE 2018-2019

Match 1 Round 6 Miscellaneous: Coordinate Geometry

1.) $y = \frac{2}{3}x + \frac{17}{3}$

2.) $y = 7x - 3$ $y = -7x + 109$

3.) $k = 7$

1.) The line $2x - 3y = C$ contains the point $(7, -1)$. Find the equation of the line parallel to this one that contains the point (C, C). Give your answer in slope-intercept form.

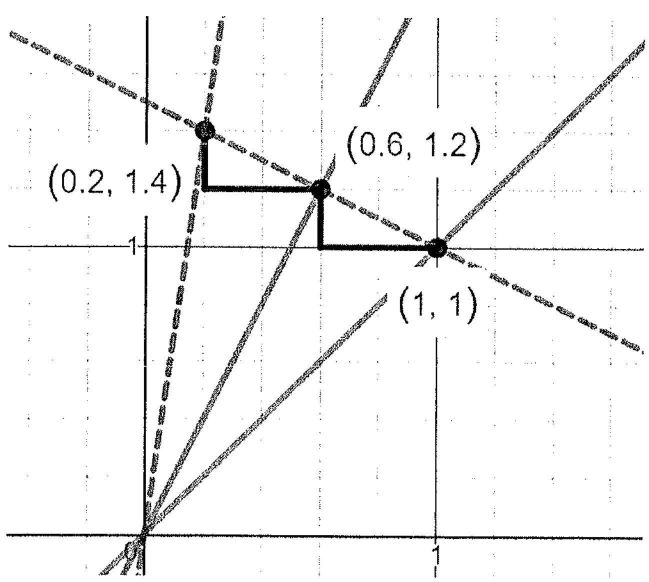
First, note that $C = 2(7) - 3(-1) = 17$. This means our new equation must be $2x - 3y = 2(17) - 3(17) = -17$, or in slope-intercept form, $y = \frac{2}{3}x + \frac{17}{3}$.

2.) Square *ABCD* has a perimeter of 40. Point *A* has coordinates (8,3) and point *C* has a *y*-coordinate of 5. Find all possible equations for the diagonal that passes through the points *B* and *D*. Give your answer(s) in slope-intercept form.

The remaining diagonal is the perpendicular bisector of the diagonal formed from the given vertices. Since the square has perimeter 40, each side has length 10. Therefore each diagonal has length $10\sqrt{2}$. We can use this information to solve for *p*: $(8-p)^2 + (3-5)^2 = (10\sqrt{2})^2 \rightarrow (8-p)^2 + 4 = 200 \rightarrow (8-p)^2 = 196 \rightarrow 8 - p = \pm 14 \rightarrow p = -6$ or $p = 22$. If $p = -6$, then the slope of the given diagonal is $-\frac{1}{7}$ and the midpoint is (1,4), making the equation of the second diagonal $y = 7x - 3$. If $p = 22$, then the slope of the given diagonal is $\frac{1}{7}$ and the midpoint is $(15,4)$, making the equation of the second diagonal $y = -7x + 109$.

3.) The line $y = 2x$ bisects the angle formed in Quadrant I by the lines $y = x$ and $y = kx$. Find the value of k.

See the diagram below. One way to solve this problem is to consider the line $y = 2x$ to be the altitude of an isosceles triangle with one side along the line $y = x$. Choose an arbitrary point (1,1) along the line $y = x$. The base of the triangle would be perpendicular to the altitude and so would have a slope of $-\frac{1}{2}$. Therefore, the equation of a base with this slope containing the point (1,1) would be $y = -\frac{1}{2}x + \frac{3}{2}$. To find where this base would intersect the altitude, set $2x = -\frac{1}{2}x + \frac{3}{2}$. This leads to the point of intersection $\left(\frac{3}{5}, \frac{6}{5}\right)$. The point of intersection between the base and the second side of the triangle can be found by using the linear pattern of the points along the base: $(1,1) \rightarrow (.6,1.2) \rightarrow (.2,1.4)$. This last point must lie on the line $y = 7x$.



Team Round

FAIRFIELD COUNTY MATH LEAGUE 2018-19 Match 1 Team Round

1.) $a = 6, a = \frac{19}{3}$

2.) $\frac{15-5\sqrt{3}}{2}$

3.) $-\frac{3}{4}$

4.) 7

5.) $(5,2), \left(-6,-\frac{5}{3}\right), \left(\frac{-1+\sqrt{41}}{2},\frac{1+\sqrt{41}}{2}\right), \left(\frac{-1-\sqrt{41}}{2},\frac{1-\sqrt{41}}{2}\right)$

6.) $\frac{312}{5}$ or 62.4.

1.) Find all values of a such that the equation $\frac{4x-2}{x+3} + 2 = \frac{ax+5}{x+3}$ has no solutions for x.

2.) A kite with perimeter 10 has a pair of opposite angles measuring 60° and 120°. Find the exact length of the smaller diagonal of the kite.

3.) The point $P(3,7)$ is reflected across the line $y = 2x + b$ to make the point P'. If P' has coordinates $(8, t)$, find $b + t$.

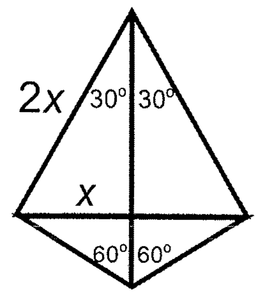
4.) Consider a set of five consecutive integers. If the smallest integer's value is decreased by k percent and the largest integer's value is increased by k percent, the sum of the five values is increased by 10 percent. If $0 < k < 100$, find the largest possible value of the median of the set. (Note: k does not have to be an integer.)

5.) Solve the system for all ordered pairs (x, y): $\begin{cases} \frac{x+1}{y} + \frac{3y}{x+1} = 4 \\ xy = 10 \end{cases}$

6.) Consider trapezoid $TRAP$ with bases $\overline{RA}$ and $\overline{TP}$ and acute angles T and P. It is known that $\cos(T) = \frac{4}{5}$, $\tan(P) = \frac{4}{3}$, the perimeter of the trapezoid is 40 and the midsegment (line segment joining the midpoints of the non-parallel sides) has a length of 13. What is the area of the trapezoid?

1.) The equation will have no solution when $x = -3$. Multiplying through by $x + 3$ yields $4x - 2 + 2(x + 3) = ax + 5 \rightarrow 6x + 4 = ax + 5$. This will have no solution if $a = 6$. Letting $x = -3$ yields $6(-3) + 4 = -3a + 5 \rightarrow -19 = -3a \rightarrow a = \frac{19}{3}$.

2.) Consider the diagram to the right. Defining half the length of the smaller diagonal as x yields sides of length $2x$, $2x$, $\frac{2\sqrt{3}}{3}x$, and $\frac{2\sqrt{3}}{3}x$. Setting the perimeter equal to 10 yields the equation $\left(4+\frac{4\sqrt{3}}{3}\right)x = 10$. Solving for x yields $x = \frac{30}{12+4\sqrt{3}} = \frac{15-5\sqrt{3}}{4}$. The length of the smaller diagonal is twice this value or $\frac{15-5\sqrt{3}}{2}$.



3.) The line of reflection is the perpendicular bisector of the line segment connecting P and P'. Therefore the line segment $\overline{PP'}$ must have a slope of $-\frac{1}{2}$. Setting up $\frac{t-7}{8-3} = -\frac{1}{2}$ yields $t = \frac{9}{2}$. The line of reflection must pass through the midpoint of $\overline{PP'}$, which is $\left(\frac{3+8}{2}, \frac{7+4.5}{2}\right)$ or $\left(\frac{11}{2}, \frac{23}{4}\right)$. Setting $\frac{23}{4} = 2\left(\frac{11}{2}\right) + b$ yields $b = -\frac{21}{4}$. Therefore $b + t = -\frac{21}{4} + \frac{9}{2} = -\frac{3}{4}$.

4.) Let n represent the smallest of the 5 integers. From the statement given in the problem, it follows that:

$$\left(1-\frac{k}{100}\right)n + n + 1 + n + 2 + n + 3 + \left(1+\frac{k}{100}\right)(n+4) = 1.1(5n+10)$$

$$5n + 10 + \frac{k}{25} = 5.5n + 11$$

$$\frac{2}{25}k - 2 = n$$

Since $0 < k < 100$, it follows that $-2 < n < 6$, so the largest possible value for the smallest integer is 5. This would correspond to the set $\{5,6,7,8,9\}$, which has a median of 7.

5.) Let $u = \frac{x+1}{y}$. The first equation then becomes $u + \frac{3}{u} = 4 \rightarrow u^2 - 4u + 3 = 0 \rightarrow (u-3)(u-1) = 0$. This yields two cases: $\frac{x+1}{y} = 3$ and $\frac{x+1}{y} = 1$.

Case 1: $\frac{x+1}{y} = 3 \rightarrow x = 3y - 1$. Substituting this into the second equation yields $y(3y-1) = 10 \rightarrow 3y^2 - y - 10 = 0 \rightarrow (y-2)(3y+5) = 0$. This gives the ordered pair solutions of $(5,2)$ and $\left(-6, -\frac{5}{3}\right)$.

Case 2: $\frac{x+1}{y} = 1 \rightarrow y = x + 1$. Substituting this into the second equation yields $x(x+1) = 10 \rightarrow x^2 + x - 10 = 0 \rightarrow x = \frac{-1\pm\sqrt{41}}{2}$. Since $y = x + 1$, we get the ordered pair solutions $\left(\frac{-1+\sqrt{41}}{2}, \frac{1+\sqrt{41}}{2}\right)$ and $\left(\frac{-1-\sqrt{41}}{2}, \frac{1-\sqrt{41}}{2}\right)$.

6.) Consider the diagram to the right. Note that $\angle T$ and $\angle P$ are acute angles. Since $3k = 4m$, it follows that $m = \frac{3}{4}k$. Let $x = RA$. This gives the equation $5k + 4k + 3\left(\frac{3}{4}k\right) + 5\left(\frac{3}{4}k\right) + 2x = 40 \rightarrow$ $15k + 2x = 40$. The midsegment length yields $13 = \frac{x+x+4k+3\left(\frac{3}{4}k\right)}{2} \rightarrow 13 = x + \frac{25}{8}k$. Subtracting twice this equation from the perimeter equation gives $\frac{35}{4}k = 14 \rightarrow k = \frac{56}{35} = \frac{8}{5}$. This gives the height of the trapezoid as $3\left(\frac{8}{5}\right) = \frac{24}{5}$. The area is the height multiplied by the midsegment length, or $13\left(\frac{24}{5}\right) = \frac{312}{5}$ or 62.4.