CSAML 2018 Solutions

April 4, 2018

Round I: Arithmetic & Number Theory

1. (1 point) If you need to mix $\frac{1}{3}$ cup of juice concentrate with $2\frac{1}{2}$ cups of water to make a certain juice drink, how much concentrate would be necessary to make 8 cups of the same juice drink?

Solution:

Set up a proportion: $\dfrac{\frac{1}{3}}{\frac{5}{2}+\frac{1}{3}} = \dfrac{x}{8} \rightarrow \dfrac{1}{3}\cdot\dfrac{6}{17} = \dfrac{x}{8} \rightarrow x = \dfrac{16}{17}$.

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2. (2 points) If A, B, C, D are positive integers and $\dfrac{55}{24} = A + \cfrac{1}{B + \cfrac{1}{C + \cfrac{1}{D}}}$,

then compute $A + B + C + D$.

Solution:

$\dfrac{55}{24} = 2 + \dfrac{7}{24} = 2 + \cfrac{1}{3 + \cfrac{3}{7}} = 2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3}}}$. Now A = 2, B = 3, C = 2, D = 3, so A+B+C+D = 10

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3. (3 points) What is the remainder when 3^{2018} is divided by 21?

Solution: Note that, mod 21, $3^1 \equiv 3, 3^2 \equiv 9, 3^3 \equiv 6, 3^4 \equiv 18, 3^5 \equiv -9 \equiv 12, 3^6 \equiv 15, 3^7 \equiv 3$.

Note that, mod 21, $3^1 \equiv 3$, $3^2 \equiv 9$, $3^3 \equiv 6$, $3^4 \equiv 18 \equiv -3$, $3^5 \equiv -9 \equiv 12$, $3^6 \equiv 36 \equiv 15$, $3^7 \equiv 3$. Following this pattern, we see that $3^1 \equiv 3^7 \equiv 3^{13} \equiv \cdots \equiv 3 \pmod{21}$. Now, the remainder when 2018 is divided by 6 is 2, so the remainder when 2017 is divided by 6 is 1.

So $3^{2017} \equiv 3 \pmod{21}$, and so $3^{2018} \equiv 9 \pmod{21}$, meaning that the remainder when 3^{2018} is divided by 21 is 9.

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Round II: Algebra I (Real numbers and no transcendental functions)

1.(1 point) Simplify if: $\dfrac{(x-y)^2-z^2}{x^2-(y-z)^2}$.

Solution: $\dfrac{(x-y+z)(x-y-z)}{(x+y-z)(x-y+z)} \rightarrow \dfrac{(x-y-z)}{(x+y-z)}$

2. (2 points) Solve for y $(b \neq 0)$: $2b(y+ab)-5a\left(b+b^2\right)=3b(ab-y)-3ab(2b+5)$.

Solution:

$\rightarrow 2by+2ab^2-5ab-5ab^2=3ab^2-3by-6ab^2-15ab$

$\rightarrow 5by=-10ab \rightarrow y=-2a$

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3. (3 points) If $x \,\Delta\, y=2x^2-4y$, determine (in simplest form) $(x \,\Delta\, 2) \,\Delta\, y$.

Solution:

$x\Delta 2=2x^2-8 \rightarrow (x\Delta 2)\Delta y=2\left(2x^2-8\right)^2-4y$

$\rightarrow 2\left(4x^4-32x^2+64\right)-4y \rightarrow 8x^4-64x^2+128-4y$

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Round III: Geometry (figures are not necessarily drawn to scale)

1. (1 point) Two similar cones have volume 54 and 128, respectively. The surface area of the smaller cone is 13.5. Find the surface area of the larger cone.

Solution:

$\dfrac{V_1}{V_2}=\dfrac{54}{128}=\dfrac{27}{64}=\dfrac{3^3}{4^3}$; $\dfrac{A_1}{A_2}=\dfrac{3^2}{4^2}=\dfrac{\frac{27}{2}}{x}$; $9x=\dfrac{27}{2}\cdot 16 \rightarrow x=24$

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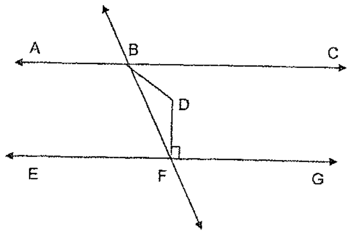
2. (2 points) In the diagram $\overleftrightarrow{BF}$ is transversal to parallel lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{EG}$. $\overline{DF} \perp \overleftrightarrow{AB}, \overline{BD}$ bisects $\measuredangle CBF$. If $m\measuredangle BDF = 115°$, find $m\measuredangle EFB$.



Solution

$x = m\measuredangle FBD, y = m\measuredangle BFD \rightarrow x + y = 65$

$2x + y + 90° = 180° \rightarrow 2x + 65 - x + 90° = 180°$

$x = 25° \rightarrow 2x = 50°$

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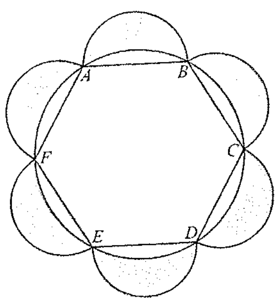
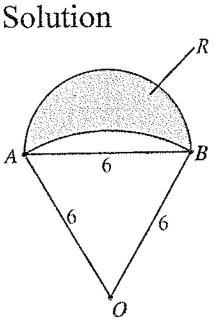
3. (3 points) Regular hexagon ABCDEF has side length 6 and is inscribed in a circle (as shown). Semicircles are constructed with AB, BC, CD, DE, EF, FA as diameters. Compute the shaded area.



Solution



Let the center of the circle be O. Area $\Delta OAB = \frac{1}{2}(6)\left(3\sqrt{3}\right) = 9\sqrt{3}$.

Area sector $OAB = \frac{1}{6}\pi(6)^2 = 6\pi$. Area semicircle $= \frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$.

Area (R) $= 9\sqrt{3} + \frac{9\pi}{2} - 6\pi = 9\sqrt{3} - \frac{3\pi}{2}$.

Required Area $6\left(9\sqrt{3} - \frac{3\pi}{2}\right) = 54\sqrt{3} - 9\pi$

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Round IV: Algebra II

1. (1 point) Let $f(x) = 3x + 2$ and $g(x) = x^2$. Solve the equation $g\left(f\left(x\right)\right) = 16$.

Solution:

$$(3x+2)^2 = 16 \rightarrow 3x+2 = \pm 4 \rightarrow \begin{cases} 3x+2=4 \\ 3x+2=-6 \end{cases} \rightarrow x = \frac{2}{3}, -2$$

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2. (2 points) Solve $e^{4x} + 16 = 13e^{2x} - 20$.

Solution: $e^{4x} - 13e^{x} + 36 = 0 \rightarrow \left(e^{2x} - 9\right)\left(e^{2x} - 4\right) = 0 \rightarrow e^{2x} = 9, e^{2x} = 4 \rightarrow e^{x} = \pm 3, e^{x} = \pm 2$

e^x must be positive, so $e^x = 3 \rightarrow x = \ln 3, e^x = 2 \rightarrow x = \ln 2$.

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3. (3 points) Solve $\log_{13}(x^3 - 1) + \log_{\frac{1}{13}}(x-1) = 1$.

Solution:

$$\rightarrow \log_{13}\left(x^3 - 1\right) + \frac{\log_{13}(x-1)}{\log_{13}\left(\frac{1}{13}\right)} = 1 \rightarrow \log_{13}\left(x^3 - 1\right) - \log_{13}(x-1) = 1$$

$$\log_{13}\frac{\left(x^3 - 1\right)}{(x-1)} = 1 \rightarrow \frac{\left(x^3 - 1\right)}{(x-1)} = 13 \rightarrow x^2 + x + 1 = 13 \rightarrow x^2 + x - 12 = 0$$

$$\rightarrow (x+4)(x-3) = 0 \rightarrow x = \cancel{-4}, 3$$

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Round V: Analytic Geometry

1. (1 point) Find the value of $\underline{k}$ so that the given lines are parallel.

$$\begin{cases} 3x - ky = -4 \\ (k-2)y - 2x = 6 \end{cases}$$

Solution:

$$\begin{cases} 3x - ky = -4 \\ (k-2)y - 2x = 6 \end{cases} \rightarrow \begin{cases} ky = 3x + 4 \\ (k-2)y = 2x + 6 \end{cases} \rightarrow \begin{cases} m = \dfrac{3}{k} \\ m = \dfrac{2}{k-2} \end{cases}$$

Since the lines are parallel, $\frac{3}{k} = \frac{2}{k-2} \rightarrow 3k - 6 = 2k \rightarrow k = 6$

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2. (2 points) Find the equation of the ellipse with foci at (1, 3) and (-1, 3) and the minor axis has length 5.

Solution: From the foci, we know that the center is at (0, 3) and the minor axis is 5 units, so 2b = 5 and 2c = 2, $c^2 = a^2 - b^2 \rightarrow 1^2 = a^2 - \left(\frac{5}{2}\right)^2 \rightarrow a^2 = \frac{29}{4}$. So, the equation is

$$\frac{4x^2}{29} + \frac{4(y-3)^2}{25} = 1 \text{ or } \frac{x^2}{\frac{29}{4}} + \frac{(y-3)^2}{\frac{25}{4}} = 1$$

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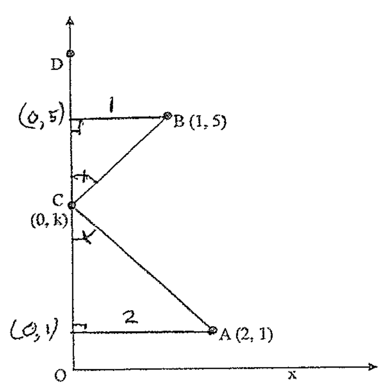
3. (3 points) Let A, B, C, D be the points shown in the diagram. Find k so that $\measuredangle BCD \cong \measuredangle ACO$.



Solution: We have two similar triangles, so

$$2(5-k) = 1(k-1) \rightarrow k = \frac{11}{3}$$

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Round VI: Trigonometry

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1. (1 point) Evaluate: $\sin 15° + \cos 15°$.

Solution:

$$(\sin 15° + \cos 15°)^2 = \sin^2 15° + 2\sin 15° \cos 15° + \cos^2 15°$$

$$= 1 + 2\sin 15° \cos 15° = 1 + \sin 30° = 1 + \frac{1}{2} = \frac{3}{2}$$

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2. (2 points) Solve for x in terms of θ (in simplest terms) if $\tan\theta(x + \cot\theta\cos\theta) = \sec\theta$.

Solution:

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$$\tan\theta\left(x+\cot\theta\cos\theta\right)=\sec\theta \to x+\cot\theta\cos\theta=\frac{1}{\cos\theta}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$\to x+\frac{\cos^2\theta}{\sin\theta}=\frac{1}{\sin\theta}\to x=\frac{1-\cos^2\theta}{\sin\theta}=\frac{\sin^2\theta}{\sin\theta}=\sin\theta$$

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3. (3 points) Let z be the complex number $\cos\theta+i\sin\theta$, where $0\le\theta<2\pi$, and suppose that $\left(z+\frac{1}{z}\right)^2=1$. Find all possible values of θ.

Solution: $z=\cos\theta+i\sin\theta$, and so, by De Moivre's Theorem, $z^{-1}\cos\left(-\theta\right)=i\sin\left(-\theta\right)$

and finally, $z^{-1}=\cos\theta-i\sin\theta$. Therefore, $z+z^{-1}=2\cos\theta$. We are given that $\left(z+z^{-1}\right)^2=1$. so

$\left(2\cos\theta\right)^2=1$, so $2\cos\theta=\pm1$, so $\cos\theta=\pm\frac{1}{2}$. The solutions to this trigonometric equation are

$\theta=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$.

Team Round April 4, 2018

1.) What 5-digit number $32a1b$ is divisible by 156? (a and b represent digits)

Solutions:

$156=(2)^2(3)(13)$, so N = 32a1b needs to be divisible by 4, 3, and 13. For N to be divisible by 4 we need 1b to be divisible by 4, so b = 2 or 6. For N to be divisible by 3 we need $3+2+a+1+b$ to be divisible by 3, so we need a + b to be divisible by 3. So if b = 2, a = 1, 4, or 7, and if b = 6, a = 0, 3, 6, or 9.
To summarize, we have (a, b) = (1, 2) or (4, 2) or (7, 2) or (0, 6) or (3, 6) or (6, 6) or (9, 6).
We have to find out which of these makes N divisible by 13.
Now the remainder when 32010 is divided by 13 is 4. So, for N to be divisible by 13, we need a0b + 4 to be divisible by 13. Checking the above seven possibilities, only (9, 6) has this property. ((9, 6) has this property, since 906 + 4 = 910 is divisible by 13.)
So N = 32916.

The number $156=2^2\cdot3\cdot13$.
We can use divisibility rules to work through the number. Since the number is divisible by 4, the last two digits must be divisible by 4, so $\underline{b}=2$ or $\underline{b}=6$. Now we can check to find some possible $\underline{a}$ values. Notice that the number is **not** divisible by 8. This means that

$\underline{a}1\underline{b} \not\equiv 8$. If $\underline{b}=2 \rightarrow a=0,2,4,6,8$ or if $\underline{b}=6 \rightarrow a=1,3,5,7,9$. Now we have only 10 numbers to check. We can eliminate a few more with divisibility by 3, (the sum of the digits must be divisible by 3) This eliminates $32,021; 32,212; 32,612; 32,812$ and $32,116; 32,516; 32,716$. Leaving us with $32,412; 32,316; 32,916$. Divide by 13 and eliminate the first two and finally get **32,916**.

Or, you can notice that $206(156)=32,136$. Now, add 156 multiple times until the tens position is 1 (one). $32,136+156=32,292; 32,292+156=32,448; 32,448+156=32,604$ $32,604+156=32,760; 32,760+156=32,916$.

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2) A student's average after 3 equally weighted test grades is 84. His first test grade is five points less than the second test grade. His third test is eight more than his second grade. How many points higher than his grade on the 3rd test would he have to score on a 4th (equally weighted test) in order to bring his average up to exactly 87?

Solution:
Let x be the Test 2 score.
$x-5+x+x+8=3(84) \rightarrow 3x=249 \rightarrow x=83$
So the Test 3 score is $83+8=91$.
Let y be the Test 4 score. We need $(3(84)+y) \div 4 = 87$, from which we get that $y=96$.
Note that $96-91=5$. So the Test 4 score needs to be 5 more than the Test 3 score.

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3) In triangle $ABC, AB=4, BC=5,$ *and* $CA=6$. Find the length of the altitude from B.

Solution: Since we are given the three sides of the triangle, use Heron's formula for the area and then $A=\frac{1}{2}bh$ to find the height.

$$A_{ABC}=\sqrt{s(s-a)(s-b)(s-c)}; s=\frac{4+5+6}{2}=\frac{15}{2}$$

$$A_{ABC}=\sqrt{\frac{15}{2}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{7}{2}\right)}=\frac{1}{4}\sqrt{15^2(7)}=\frac{15}{4}\sqrt{7}$$

$$\rightarrow$$

$$A_{ABC}=\frac{1}{2}bh \rightarrow \frac{15\sqrt{7}}{4}=\frac{1}{2}(6)h$$

$$h=\frac{15\sqrt{7}}{4(3)}=\frac{5\sqrt{7}}{4}$$

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4)) Given: $f(x)=\ln\left(e^{2x}+2e^{x}\right)$. Write an expression for $f^{-1}(x)$ in terms of x.

Let $y = f(x) \rightarrow y = \ln\left(e^{2x} + 2e^{x}\right)$. Now,

$$e^{y} = e^{2x} + 2e^{x} = \left(e^{x} + 1\right)^{2} - 1$$

$$e^{y} + 1 = \left(e^{x} + 1\right)^{2} \rightarrow e^{x} = \sqrt{e^{y} + 1} - 1$$

$$x = \ln\left(\sqrt{e^{y} + 1} - 1\right) \rightarrow f^{-1}(x) = \ln\left(\sqrt{e^{x} + 1} - 1\right)$$

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5) Find the radius of the circle that has the same center as $x^2 + y^2 - 6x + 2y = 0$ and is tangent to the line $x - y - 1 = 0$.

Solution: Or find the distance from the center of the circle to the line. Find the center of the circle by completing the square:

$$x^2 - 6x + 9 + y^2 + 2y + 1 = 10 \rightarrow (x-3)^2 + (y+1)^2 = 10 \rightarrow \text{center} = (3, -1)$$

Now find the distance from (3, -1) to

$$x - y - 1 = 0 .\ d = \frac{\left|Ax_1 + By_1 + C\right|}{\sqrt{A^2 + B^2}} = \frac{\left|1(3) + (-1)(-1) + (-1)\right|}{\sqrt{(1)^2 + (-1)^2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

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6) If $\log_{\cos x} \sin x = \frac{1}{2}$, find the exact value of $\sec x$.

Solution:

$$\log_{\cos x} \sin x = \frac{1}{2} \rightarrow \sqrt{\cos x} = \sin x \rightarrow \cos x = 1 - \cos^2 x \rightarrow \cos^2 x + \cos x - 1 = 0$$

$$\cos x = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{-1 + \sqrt{5}}{2} \rightarrow \sec x = \frac{1}{\cos x} = \frac{2}{-1 + \sqrt{5}}\left(\frac{-1 - \sqrt{5}}{-1 - \sqrt{5}}\right) = \frac{1 + \sqrt{5}}{2}$$