FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 6 Round 1
Geometry: Lines and Angles

1.) ________210_____________

2.) __________-10___________

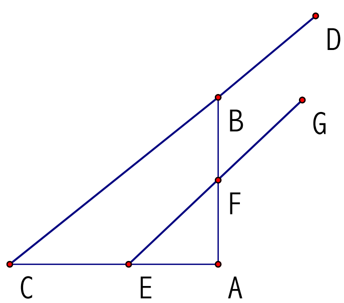
3.) ________25, 3_____________

Note: Diagrams are not necessarily to scale:

1) The supplement of angle X measures 30 degrees less than three times its complement. What is the sum in degrees of the measures of the complement and supplement of angle X?

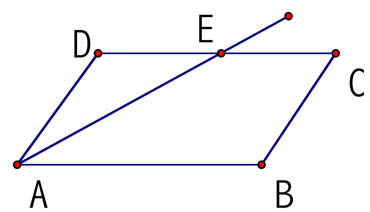
   Solve 180-x=3(90-x)-30, so 180-x=270-3x-30, so 2x=60, and x=30. The complement of x measures 60, and the supplement of x measures 150, so their sum is 210.

2.) ΔAFE is a right triangle with the right angle at A. The measure of ÐFEA is (6y+130) degrees, and the measure of ÐDBF is (120-4y). What is the value of y that would make $\overleftrightarrow{CD}$ parallel to $\overleftrightarrow{EG}$?



ÐFEA is complementary to ÐEFA, ÐEFA=ÐBFG by vertical angles, and if the two lines are parallel, then ÐBFG is supplementary to ÐDBF, so 90-(6y+130)=180-(120-4y), so -6y-40=60+4y, so 10y=-100, so y=-10.

3.) ABCD is a parallelogram. $\overrightarrow{AE}$ bisects and intersects $\overline{CD}$ at point E. The measure of ÐADE is z^2+30, and the measure of ÐEAB is z+15. What are all possible values for the measure of ÐDEA in degrees?



$\angle EAB$ is congruent to $\angle DEA$ by alternate interior angles, and since ray AE is a bisector, the measure of $\angle EAB$ is the same as the measure of $\angle DAE$. All the angles in ΔADE add to 180 degrees, so $z^2+30+(z+15)+(z+15)=180$, so $z^2+2z-120=0$, so $(z+12)(z-10)=0$, so $z=-12$ or $x=10$. Neither gives extraneous answers, so the measure of $\angle DEA$ could be $10+15 = 25$ degrees or $-12+15= 3$ degrees.

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Match 6 Round 2
Algebra: Literal Equations

1.) ______y= $\frac{-4}{3}x+\frac{19}{3}$________

2.) ______ $z=\frac{-9}{4p},\frac{1}{p},\frac{9}{4p},\frac{-1}{p}$__

3.) ______ $a=\pm\sqrt{-3b}$ ______________

1) If $t=\frac{3-y}{4}$ and $t=\frac{2x-5}{6}$, express y in terms of x using the form y=mx+b for constants m and b.

-y=4t-3, so y=-4t+3. Y=-4($\frac{2x-5}{6}$)+3 = = $\frac{-4}{3}x+\frac{19}{3}$

2) If p≠0, solve for all possible expressions of z in terms of p and simplify as much as possible:

$$(4p^2z^2-9)^2-25p^2z^2=0$$

This is a difference of squares, so we have

$$(4p^2z^2-9+5pz)(4p^2z^2-9-5pz)=0$$

$$(4pz+9)(pz-1)(4pz-9)(pz+1)=0$$

$$4pz+9=0, pz-1=0, 4pz-9=0, pz+1=0$$

$$z=\frac{-9}{4p},\frac{1}{p},\frac{9}{4p},\frac{-1}{p}$$

3) If a≠1 and b<0, solve for all possible expressions of a in terms of b and simplify as much as possible:

$$5a^3-6(a+b)(a-2b)-12b^2=a^2(a-2)-6ab+12b$$

$$5a^3-6(a^2-ab-2b^2)-12b^2=a^3-2a^2-6ab+12b$$

$$5a^3-6a^2+6ab+12b^2-12b^2=a^3-2a^2-6ab+12b$$

$$4a^3-4a^2+12ab-12b=0$$

$$4a^2(a-1)+12b(a-1)=0$$

$$(4a^2+12b)(a-1)=0$$

$$a \neq 1,$$

$$a^2=-3b, a=\pm\sqrt{-3b}$$

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Match 6 Round 3 Geometry: Solids and Volumes

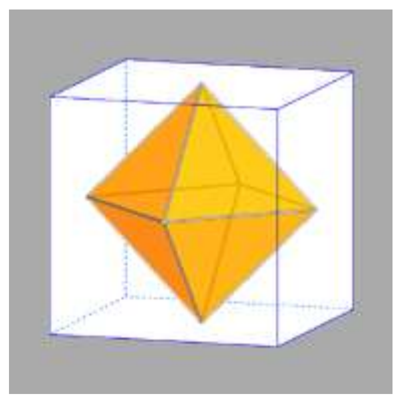
1.) ________ $\frac{64p}{3}$ ___________

2.) ________ $16\sqrt{3}$ ___________

3.) _______ $\frac{12 - 4\sqrt[3]{9}}{3}$ __________

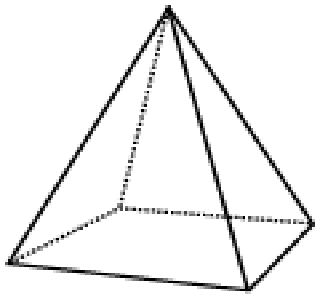
1) A spherical scoop of ice cream of radius 4 cm is placed on in a right circular cone of height 12 cm and base radius 4 cm so that half of the ice cream scoop remains outside of the cone. Find the volume of the cone that is not taken up by the ice cream.

The ice cream has volume $\frac{1}{2}*\frac{4}{3}p*4^3 = \frac{256p}{6} = \frac{128p}{3}$ The cone has volume $\frac{1}{3}p*12*4^2 = 64p$. $64p - \frac{128p}{3} = \frac{192p}{3} - \frac{128p}{3} = \frac{64p}{3}$



2) An octahedron is placed inside a cube so that the 6 vertices of the octahedron meet the centers of each of the 6 sides of the cube. If the cube has volume 64 cm^3, what is the surface area of the octahedron?

The octahedron consists of 2 square pyramids. If the cube has volume 64 cm^3, each side of the cube is 4 cm, so the distance from one point to the other on the base of the octahedron is $2\sqrt{2}$ cm, and this is the length of each side of the octahedron. To find the slant height of each of the triangles, they would have to be x, where $x^2 = (2\sqrt{2})^2$ (the length of one of the sides of the octahedron) $-\left(\sqrt{2}\right)^2$, ($\sqrt{2}$ is half of one of the sides of the octahedron), so, so x= $\sqrt{8 - 2} = \sqrt{6}$. The octahedron consists of 8 triangles with base $2\sqrt{2}$ and height $\sqrt{6}$, so $8*\frac{1}{2}(2\sqrt{2})\sqrt{6} = 8\sqrt{12} = 16\sqrt{3}$



3) A square pyramid has its base area 16 cm^2 and height 4 cm. A plane parallel to the base is passed through the pyramid so that the volume of the pyramid above the plane is $\frac{1}{3}$ of the volume of the original pyramid. How many cm above the base is the plane? Express your answer as a single fraction.

The volume of the original pyramid is $\frac{1}{3} * 4^2 * 4 = \frac{64}{3}$. If the plane passes at z=k, each side of the base of the new pyramid will vary linearly with k, since the sides of the pyramid are lines. As k goes from 0 to 4, each base goes from 4 to 0, so each base will be (4-k). The height of the new pyramid will also be (4-k) by similarity, so we have

$\frac{1}{3}(4 - k)^3 = \frac{1}{3} * \frac{64}{3}$, so $(4 - k)^3 = \frac{64}{3}$, so $4 - k = \sqrt[3]{\frac{64}{3}}$, and k= $4 - \sqrt[3]{\frac{64}{3}} =$
$4 - \frac{4}{\sqrt[3]{3}} = 4 - \frac{4\sqrt[3]{9}}{3} = \frac{12 - 4\sqrt[3]{9}}{3}$

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Match 6 Round 4
Radical Expressions and Equations

1.) $\dfrac{65\sqrt[3]{3}}{3}$

2.) 2

3.) 4

1) Express in simplest radical as an integer or reduced fraction:

$$4\sqrt[3]{375}+5\sqrt[3]{\frac{1}{9}}$$

$$4\sqrt[3]{375}+5\sqrt[3]{\frac{1}{9}}=4\sqrt[3]{125}\sqrt[3]{3}+5\sqrt[3]{\frac{3}{27}}=20\sqrt[3]{3}+\frac{5}{3}\sqrt[3]{3}=$$

$$\frac{60}{3}\sqrt[3]{3}+\frac{5}{3}\sqrt[3]{3}=\frac{65\sqrt[3]{3}}{3}$$

2) What is the value of $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$?

Let x = $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$. Then x=$\sqrt{2+x}$. Square both sides to get x^2 = x+2, so x^2-x-2=0, and x= (x-2)(x+1)=0. It is clearly not -1, so x=2.

3) Solve for all positive real values of x: $\sqrt{2x+10}-\sqrt{2}=\sqrt{\frac{x+12}{2}}$

It would be difficult to check the negative answer, so use only the positive answer.

Rewrite this as $\sqrt{2}\sqrt{x+5}-\sqrt{2}=\frac{\sqrt{x+12}}{\sqrt{2}}$, so that $2\sqrt{x+5}-2=\sqrt{x+12}$. Square both sides to get $4(x+5)-8\sqrt{x+5}+4=x+12$, so that $3x+12=8\sqrt{x+5}$ and square both sides again to get $9x^2$ + 72x + 144 = 64x + 320, so $9x^2$ + 8x - 176=0, which factors to (x-4)(9x+44)=0, so x=4, which checks since √18 - √2 = √8, 3√2 - √2 = 2√2.

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Match 6 Round 5
Polynomials and
Advanced
Factoring

1.) __________________-53__________________

2.)______________ $-1, 2, \pm\sqrt{3}$ __________________

3.) _______ $(x^2+3y^2+2xy)(x^2+3y^2-2xy)$ ____

1.) What is the remainder when $4x^3-21x^2+18x-26$ is divided by x^2-6x+9 ?

When you divide $4x^3-21x^2+18x-26$ by x^2-6x+9, you get 4x+3 with remainder -53

2.) Find the four real zeros of $x^4-x^3-5x^2+3x+6$

By the Integer Zero Theorem, the only possible integer zeros are ±1,±2,±3,±6. Checking some of them, we find that -1 and 2 yield zero when substituted for x. Dividing sequentially by x+1 and x-2 yields x^2-3 as the depressed polynomial, so the other two zeros are ±√3.

3) Factor $x^4+2x^2y^2+9y^4$ as a product of 2 polynomials with integer coefficients.

$$x^4+2x^2y^2+9y^4 = x^4+6x^2y^2+9y^4-4x^2y^2 =$$

$$(x^2+3y^2)^2-(2xy)^2 = (x^2+3y^2+2xy)(x^2+3y^2-2xy)$$

FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 6 Round 6
Counting and Probability

1.) ______1,3,5,7_______________

2.) _________64____________

3.) _________ $\frac{131}{210}$ ______________

1) If r is an integer such that $0 \le r \le 8$, for which values of r is the expression $\frac{8!}{r!(8-r)!}$ a multiple of 8?

The row beginning with 1 8 of Pascal's triangle goes 1 8 28 56 70 56 28 8 1.
The only multiples of 8 are 8 and 56, and these correspond with r=1,3,5,7

2) At Pepe's Pizza, you can get any of the following items on your pizza: Pepperoni, Sausage, Onions, Anchovies, Mushrooms, or Peppers. If the order in which the items are put on the pizza is not important and you can chose any number of toppings from 0 to 6, how many different kinds of pizza can be made?
For each topping, you may either choose to have it or not. There are 2 choices for each topping, YES or NO. Multiply 2*2*2*2*2*2 = 64

3) 3 red balls, 4 white balls, and 3 green balls are placed in a container. You randomly draw 4 balls. What is the probability that you draw at least 2 red balls or at least 2 green balls? (This is not an exclusive or, so this event includes the event of drawing 2 red balls and 2 green balls.)

P(exactly 2 red) = $\frac{C(3,2)*C(7,2)}{C(10,4)} = \frac{3*21}{210} = \frac{63}{210}$.

P(exactly 3 red) = $\frac{C(3,3)*C(7,1)}{C(10,4)} = \frac{7}{210}$. The situation is the same with exactly 2 green or exactly 3 green, but we've counted the event (exactly 2 green and exactly 2 red) twice, so we need to subtract this off once, and this is $\frac{C(3,2)*C(3,2)}{C(10,4)} = \frac{9}{210}$.. Probability is

$$\frac{63}{210} + \frac{7}{210} + \frac{63}{210} + \frac{7}{210} - \frac{9}{210} = \frac{131}{210}$$

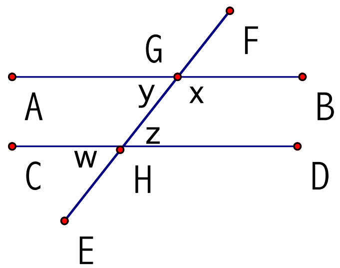
FAIRFIELD COUNTY MATH LEAGUE (FCML) 2013-2014

Match 6 Team Round

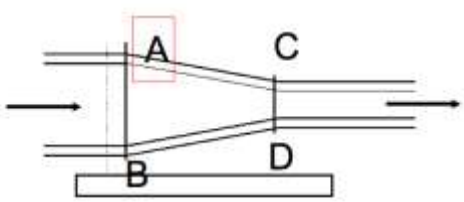
1.) _____180+8m+20n_____ 4.) _____$(a-b)^3(2a+b)^2$_____

2.) _________ $\frac{37}{2}p$_______ 5.) ______$\frac{23}{108}$____________

3.) ____$y=-1+x, y=-1-x$____ 6.) _____24___________



1) If $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ above and $\angle FGB$ = (2m+5n) °, and the angles shown are measured in degrees, express x-2y+3z+4w in terms of m and n.
Solutions: x=(180-2m+5n), while y, z, and w are all 2m+5n.
(180-(2m+5n))-2(2m+5n)+3(2m+5n)+4(2m+5n)= 180-(2m+5n)+5(2m+5n)= 180+8m+20n



2) Water flows through a pipe with a circular cross-sectional area as shown above. The diameter $\overline{AB}$ at the circular cross section to the left in the picture is 4 meters, and the diameter $\overline{CD}$ is 3 meters. $\overline{AC}$ and $\overline{BD}$ are line segments and the distance between the centers of the two circular cross-sections shown is 6 meters. Find the volume of the part of the pipe between the circle with diameter $\overline{AB}$ and the circle with diameter $\overline{CD}$.
Consider the desired volume to be the volume of a frustum of a cone. If the cone were extended to a point on the right side of the picture, the point would be 24 m to the right of $\overline{AB}$ by similarity. Find the volume of the cone that has $\overline{AB}$ as its base, and subtract off the volume of the cone that contains $\overline{CD}$ as its base. So we have

$$\frac{1}{3}p(2^2)(24) - \frac{1}{3}p(1.5^2)(18)$$
$$= \frac{96}{3}p - \frac{40.5}{3}p = \frac{55.5}{3}p = \frac{111}{6}p$$
$$= \frac{37p}{2}$$

3.) .) If -1<x<1, solve for y in terms of x:. $y + 2 = \sqrt{2y + x^2 + 3}$ Square both sides to get

$y^2 + 4y + 4 = 2y + x^2 + 3$

$y^2 + 2y + (1 - x^2) = 0$

$y = \frac{-2 \pm \sqrt{(-2)^2 - 4(1 - x^2)}}{2} = \frac{-2 \pm \sqrt{4x^2}}{2} = \frac{-2 \pm 2|x|}{2} = -1 \pm x$. Check for extraneous solutions: If y=-1+x, then $(-1 - x) + 2 = \sqrt{2(-1 - x) + x^2 + 3}$ and $\sqrt{x^2 + 2x + 1} = x + 1$, which is true only if x>-1, so this is OK. If y=-1-x, $(-1 - x) + 2 = \sqrt{2(-1 - x) + x^2 + 3}$, and $\sqrt{x^2 - 2x + 1} = -x + 1$, which is true if x<1, so this is also OK. So the answers are y=-1+x and y=-1-x

4.) Factor into 5 polynomials with integer coefficients:

$4a^5 - 8a^4b + a^3b^2 + 5a^2b^3 - ab^4 - b^5$

If a=b, this adds to zero, so a-b is a factor. Divide by a-b to get

$4a^4 - 4a^3b - 3a^2b^2 + 2ab^3 + b^4$

Again, you find that a-b is a factor, so divide by a-b to get

$4a^3 - 3ab^2 - b^3$, and then a-b is again a factor, so divide by a-b to get

$4a^2 + 4ab + b^2$, which factors to $(2a+b)^2$, so the complete factoring is

$(a - b)^3(2a + b)^2$

5.) In the game of Yahtzee, 5 regular six-sided dice with sides labeled 1,2,3,4,5, and 6 are rolled at each turn. What is the probability that for a single turn at least 3 dice show the same number? We want p(exactly 3) + p(exactly 4) + p(exactly 5). Suppose the two other dice show the same number. There are ${}_5C_3$ ways to choose locations for the three and the two. There are 6 choices for the number that appears 3 times, and then 5 choices for the number that appears twice. If the two numbers on the other two dice are different, then it's the same, except multiply by 5*4 instead of 5. To find p(exactly 4), there are 6 numbers that can show 4 times, ${}_5C_4$ locations of dice for that number to show, and then 5 numbers that can appear on the single die. There are only 6 ways of all 6 dice showing the same number, so the total is

$$\frac{6*{}_5C_3*5}{6^5}+\frac{6*{}_5C_3*5*4}{6^5}+\frac{6*{}_5C_4*5}{6^5}+\frac{6}{6^5}=$$

$$\frac{{}_5C_3*5}{6^4}+\frac{{}_5C_3*5*4}{6^4}+\frac{{}_5C_4*5}{6^4}+\frac{1}{6^4}=$$

$$\frac{50}{1296}+\frac{200}{1296}+\frac{25}{1296}+\frac{1}{1296}=\frac{276}{1296}=\frac{23}{108}$$

6.) In the choir, there are N sopranos, N-3 altos, N-1 tenors, and N-4 basses. They arrange themselves in a line. The members of each group may stand in any order, but all the members of each group must all stay in their section.. Let W_S, W_A, W_T, and W_B be the number of ways the sopranos, altos, tenors, and basses can be arranged. If the product of W_A and W_S is 40 times the product of W_T and W_B, how many members are in the choir?

We have W_S = N! W_A= (N-3)!, W_T=(N-1)!, and W_B=(N-4)!, so that N!(N-3)! = 40(N-1)!(N-4)!

Since $\frac{N!}{(N-1)!}$ = N, and $\frac{(N-3)!}{(N-4)!}$ = N-3, we have N(N-3)=40, so $N^2 - 3N - 40 = 0$, so (N-8)(N+5)=0, so N=8. 8+5+7+6 = 24