

FAIRFIELD COUNTY MATH LEAGUE 2025–2026

Match 6

Individual Section

Please write your answers on the answer sheet provided.

Round 1: Lines and Angles

- 1-1 The sum of the measures of the interior angles of a regular convex polygon is four times the sum of its exterior angles. What is the difference between the measures of one interior angle and one exterior angle?
[Answer: 108]

Since the sum of the exterior angles must be 360° , we have $180(n - 2) = 4 * 360$, which is solved to yield $n = 10$. The measure of one interior angle of a decagon is $180 - \frac{360}{10} = 144$, making one exterior angle measure 36, and therefore the desired quantity is $144 - 36 = 108$.

- 1-2 A lattice point has integer coordinates. The corners of triangle ABC are lattice points. C is at the origin, B is on the positive x -axis, and A is in the first quadrant. The slope of \overrightarrow{AC} is $\frac{5}{3}$. The slope of \overrightarrow{AB} is $-\frac{3}{4}$. What is the minimum sum of A 's coordinates?
[Answer: 24]

Since the numerators of the slopes can both represent the distance from A to the x -axis, we must write them with the least common multiple. Doing so gives us slopes of $\frac{15}{9}$ and $-\frac{15}{20}$, and these values would give A coordinates of $(9,15)$, making the desired quantity $9 + 15 = 24$.

- 1-3 A certain convex polygon contains no right angles, and its number of obtuse angles is 2026 times its number of acute angles. Find the greatest possible number of sides of the polygon.
[Answer: 6081]

The greatest number of acute angles that a convex polygon can have is three. An easy demonstration of this is that an acute interior angle will have an obtuse exterior angle, and more than three such exterior angles would add up to more than 360° . Therefore, the maximum number of sides of the polygon must be $2026(3) + 3 = 6078 + 3 = 6081$.

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Round 2: Literal Equations

- 2-1 Consider the simultaneous equations $a = \frac{1}{b} + 2$, $b = \frac{c+6}{c+1}$, and $c = a - 4$. The possible values of a are $\frac{m \pm \sqrt{n}}{p}$, where n has no perfect square factors greater than 1. Find the product mnp .

[Answer: 10]

Substituting $c = a - 4$ into the equation for b yields $b = \frac{a+2}{a-3}$. Substituting this into the equation for a yields $a = \frac{a-3}{a+2} + 2$, which after multiplying by $a + 2$ and collecting terms yields $a^2 - a - 1 = 0$. This means $a = \frac{1 \pm \sqrt{5}}{2}$, making the desired quantity $(1)(5)(2) = 10$.

- 2-2 If $3fg + fg^2 + 48 = 3g^2$ for positive real numbers f and g , then f can take any value in the interval $(0, a)$ and the smallest possible integer value of g is b . Find b^a .

[Answer: 125]

Isolating f in terms of g yields $f = \frac{3g^2 - 48}{g^2 + 3g} = \frac{3(g+4)(g-4)}{g(g+3)}$. Note for f to be positive, g must be greater than 4, so $b = 5$. Additionally since $f = 3 \left(\frac{(g+4)(g-4)}{g(g+3)} \right)$ and $\frac{(g+4)(g-4)}{g(g+3)} < 1$ for all $g > 4$, so the range of f is $(0, 3)$, and therefore the desired quantity is $5^3 = 125$.

- 2-3 Every ordered pair (A, B) that satisfies the equation $y = \frac{2+3x}{5+x^2}$ also satisfies the equation $(2xy - a)^2 = (b + cy)(d - ey)$ for positive integers a, b, c, d , and e . Find $a + b + c + d + e$.

[Answer: 25]

Multiplying both sides by $x^2 + 5$ yields $x^2y + 5y = 2 + 3x$, and multiplying by $4y$ and collecting terms yields $4x^2y^2 - 12xy = 8y - 20y^2$. Since $(2xy - 3)^2 = 4x^2y^2 - 12xy + 9$, we have $4x^2y^2 - 12xy + 9 = 9 + 8y - 20y^2$, which is equivalent to $(2xy - 3)^2 = (1 + 2y)(9 - 10y)$, making the desired quantity $3 + 1 + 2 + 9 + 10 = 25$.

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Round 3: Solids & Volume

- 3-1 Phil and Bobby each have a piece of 8" x 12" paper. Each rolled their paper to make a cylinder. Phil rolled his to make a tall, narrow cylinder, while Bobby rolled his to make a shorter, wider cylinder. The difference in the volumes of their cylinders is $\frac{a}{\pi}$ cubic inches.

Find a .

[Answer: 96]

One cylinder will have a height of 8 and a base radius of $\frac{6}{\pi}$, while the other will have a height of 12 and a radius of $\frac{4}{\pi}$. The two respective volumes are $\pi \left(\frac{6}{\pi}\right)^2 (8) = \frac{288}{\pi}$ and $\pi \left(\frac{4}{\pi}\right)^2 (12) = \frac{192}{\pi}$, which have a difference of $\frac{96}{\pi}$, making the desired quantity 96.

- 3-2 A composite figure is created as follows. The sculptor begins with a solid prism with 10 cm by 10 cm square base and 20 cm height, then carves a piece out of the bottom and adds it to the top. She removes a right square pyramid with height 12 cm and a base that perfectly aligns with the bottom base of the prism. Then she attaches the pyramid to the top of the prism so that there is no overlap. What is the surface area of the composite figure in square centimeters?

[Answer: 1320]

The resulting figure will have a surface area equal to the lateral surface area of the prism plus twice the lateral surface area of the pyramid. We can use the Pythagorean theorem to find that each triangular face of the pyramid will have a height of $\sqrt{5^2 + 12^2} = 13$ cm. The lateral surface area of the prism is $4(10)(20) = 800$ and the lateral surface of the pyramid is $4\left(\frac{1}{2}\right)(10)(13) = 260$, making the desired quantity $800 + 2(260) = 1320$.

- 3-3 A particular solid has surface area S and volume V . If a similar solid has surface area $2S$ and volume $V + 10$, then $V = \frac{a+b\sqrt{c}}{d}$ where a, b, c , and d are positive integers, c has no perfect square factors greater than 1, and a, b , and d have no common factors greater than 1. Find $a + b + c + d$.

[Answer: 39]

A similar solid with twice the surface area has a side length ratio with the smaller solid of $\sqrt{2}:1$, making the volume ratio $2\sqrt{2}:1$. This means $2\sqrt{2}V = V + 10$, and squaring both

sides yields $8V^2 = V^2 + 20V + 100$, or $7V^2 - 20V - 100 = 0$, which when solved using the quadratic formula yields $\frac{20 \pm \sqrt{400 + 2800}}{14} = \frac{20 \pm 40\sqrt{2}}{14} = \frac{10 \pm 20\sqrt{2}}{7}$, and as the negative quantity is extraneous, the desired quantity is $10 + 20 + 2 + 7 = 39$.

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Round 4: Radical Expressions and Equations

- 4-1 Let $A = \sqrt{2026 + \sqrt{2026 + \sqrt{2026 + \dots}}}$ and $B = \sqrt{2026 - \sqrt{2026 - \sqrt{2026 - \dots}}}$
Find $A - B$.
[Answer: 1]

Note $A^2 = 2026 + \sqrt{2026 + \sqrt{2026 + \dots}} = 2026 + A$, and likewise $B^2 = 2026 - B$,
Therefore $A^2 - B^2 = A + B$, and consequently $(A + B)(A - B) = A + B$, meaning
 $A - B = 1$.

- 4-2 Find the sum of all real values of x that satisfy the equation $\sqrt[3]{x^2 - 4} = \frac{x+2}{\sqrt[3]{x+7}}$.
[Answer: 16]

Multiplying both sides by $\sqrt[3]{x+7}$ yields $\sqrt[3]{x+7}\sqrt[3]{x^2-4} = x+2$. Squaring both sides
yields $(x+7)(x^2-4) = (x+2)^3$, or $(x+7)(x-2)(x+2) = (x+2)^3$. Note
immediately that $x = -2$ is a solution to this equation. Next we can divide both sides by
 $x+2$ to get $(x+7)(x-2) = (x+2)^2$, or $5x^2 - 14 = 4x + 4$, yielding $x = 18$, making
the desired quantity $-2 + 18 = 16$.

- 4-3 Consider the function $f(x) = 3\sqrt{x+6} + 8$. The function $g(x) = af(x+b) + c$, where
 a, b , and c are integer constants, has a domain of $[3, \infty)$, has a range of $[-1, \infty)$, and
contains the point $(4,5)$. If $f(x)$ and $g(x)$ both contain the point (m,n) , find $m+n$.
[Answer: 42]

To match the domain and range provided, $g(x) = a\sqrt{x-3} - 1$, and we can find the value
of a given $5 = a\sqrt{4-3} - 1$, so $a = 6$. Therefore we have $3\sqrt{x+6} + 8 = 6\sqrt{x-3} - 1$,
or $\sqrt{x+6} + 3 = 2\sqrt{x-3}$. Squaring both sides yields $x + 6 + 9 + 6\sqrt{x+6} = 4x - 12$, or
 $2\sqrt{x+6} = x - 9$. Finally squaring and rearranging this equation yields
 $x^2 - 22x + 57 = 0$, which yields $x = 19$ or $x = 3$. The latter value is extraneous, and
since $f(19) = 23$, the desired quantity is $19 + 23 = 42$.

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Round 5: Polynomials and Advanced Factoring

- 5-1 The functions $f(x) = x^3 - 4x$ and $g(x) = 2x^2 - 8$ have the points (a, b) and (c, d) in common. Find $a^2 + b^2 + c^2 + d^2$.

[Answer: 8]

Setting $x^3 - 4x = 2x^2 - 8$ yields $x^3 - 2x^2 - 4x + 8 = 0$, which factors into $(x^2 - 4)(x - 2) = (x - 2)^2(x + 2) = 0$, yielding solutions of $x = 2$ and $x = -2$, and $f(2) = f(-2) = 0$, making the desired quantity $2^2 + 0^2 + (-2)^2 + 0^2 = 8$.

- 5-2 The polynomial $k(x) = x^4 + ax^3 + bx^2 + cx + 13$ with all rational coefficients has a zero of $2 + 3i$, and the sum of all zeros is 4. Find $a + b + c$.

[Answer: 6]

Note that another zero must be $2 - 3i$, and $(2 - 3i)(2 + 3i) = 13$ and $2 - 3i + 2i + 3i = 4$. This means the remaining two zeros must have sum of 0 and a product of 1, which would be solutions to $x^2 + 1 = 0$. This makes $k(x) = (x^2 - 4x + 13)(x^2 + 1) = x^4 - 4x^3 + 14x^2 - 4x + 13$, making the desired quantity $-4 + 14 - 4 = 6$.

- 5-3 A particular cubic polynomial $g(x)$ with rational coefficients has a zero of $1 - 3i$ and a remainder of 162 when divided by $x - 4$. If one of the zeros is a positive integer, find the largest possible positive value of the leading coefficient.

[Answer: 9]

There must be another zero of $1 + 3i$ if the coefficients are rational. Knowing one of the zeros is a positive integer k gives a tentative equation of $k(x) = a(x - k)(x^2 - 2x + 10)$. By the remainder theorem, we know $k(4) = 162$, so $162 = a(4 - k)(18)$, which means $9 = a(4 - k)$. The value of a will be maximized when $k = 3$, which will make $a = 9$.

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Round 6: Counting and Probability

- 6-1 If A , B , and C are independent events occurring with probabilities 1, 0.3, and 0.8 respectively, the probability that exactly 2 of those events occur can be expressed as a reduced fraction $\frac{a}{b}$. Find $a + b$.

[Answer: 81]

Since event A must occur, the probability that exactly two events occur is $P(A \cap B \cap C^c) + P(A \cap C \cap B^c)$, and since the probabilities are independent, this computation is $(1)(.3)(.2) + (1)(.8)(.7) = .06 + .56 = .62 = \frac{31}{50}$, making the desired quantity $31 + 50 = 81$.

- 6-2 Arrange six consecutive integers from 2021 to 2026 in random order. The product of the first three numbers is p_1 , and the product of the remaining three numbers is p_2 . The probability that $p_1 + p_2$ is odd can be represented as $\frac{a}{b}$, where a and b are mutually prime. Find $a + b$.

[Answer: 11]

In order for $p_1 + p_2$ to be odd, exactly one of the two products must be odd, which means it must be the product of 3 odd numbers. The arrangements can take the form of three odd followed by three even or vice versa out of $6!$ possible arrangements. Any group of three numbers has $3! = 6$ arrangements, making the probability $2 * \frac{3!*3!}{6!} = 2 * \frac{6*6}{720} = \frac{72}{720} = \frac{1}{10}$, making the desired quantity $1 + 10 = 11$.

- 6-3 Consider independent events A , B , and C , each of which can occur with the same probability p . If $P(A \cup B \cup C)$ is 25% greater than $P(A \cup B)$, then $p = \frac{a-\sqrt{b}}{c}$, where a , b , and c are positive integers, b contains no perfect square factors greater than 1, and a and c are mutually prime. Find $a + b + c$.

[Answer: 32]

Note that $P(A \cup B) = P(A) + P(B) - P(A \cap B) = p + p - p^2 = 2p - p^2$, and $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) = p + p + p - p^2 - p^2 - p^2 + p^3 = 3p - 3p^2 + p^3$. (Alternatively, each probability is the complement of none of the events occurring, or $1 - (1 - p)^2$ and $1 -$

$(1 - p)^3$, respectively). Setting $\frac{2p-p^2}{3p-3p^2+p^3} = \frac{4}{5}$ yields $10p - 5p^2 = 12p - 12p^2 + 4p^3$, or $4p^2 - 7p + 2 = 0$. Therefore $p = \frac{7 \pm \sqrt{49-32}}{8}$, but the larger answer is extraneous (as it is larger than 1) and so $p = \frac{7-\sqrt{17}}{8}$, making the desired quantity $7 + 17 + 8 = 32$.

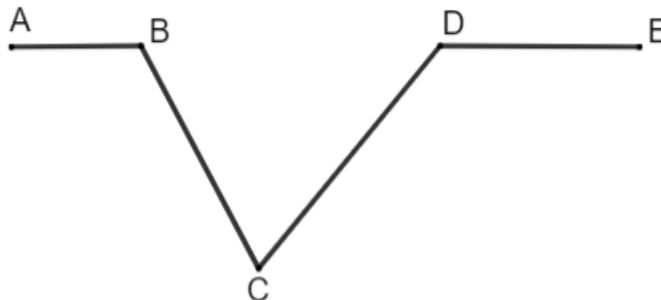
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Team Round

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- T-1 Consider the diagram (not drawn to scale). Points $A, B, D,$ and E are collinear. If $m\angle ABC, m\angle BCD,$ and $m\angle CDE$ in degrees (all greater than 0 and less than 180) form an arithmetic sequence of integers (not necessarily in that order), find the sum of the least and greatest possible values of the measure of the largest of the three angles in degrees.



[Answer: 300]

Because $m\angle CBD + m\angle BDC + m\angle BCD = 180$ and therefore $180 - m\angle ABC + 180 - m\angle CDE + m\angle BCD = 180$, we have $180 = m\angle ABC + m\angle CDE - m\angle BCD$. Note that $m\angle BCD$ must be the smallest value in order for the measures of all three to be under 180° ; otherwise you have $a + a + d - (a + 2d) = a - d = 180$ (all angles have measures greater than 180) or $a + a + 2d - (a + d) = a + d = 180$ (the middle and largest have measures 180 or greater). Therefore we have $a + d + a + 2d - a = a + 3d = 180$. The values we seek are found by maximizing either a or d , which minimizes the other value. The maximum value of a occurs when $d = 1$, yielding $a = 177$, and making the largest angle measure 179. The minimum a occurs when $d = 59$, yielding $a = 3$ and making the largest angle measure 121. The desired quantity is therefore $179 + 121 = 300$.

- T-2 On the xy -plane, the equation $2x^3 + 2xy - 8x^2 - 6x - 8y + 24 = 0$ describes a line and a parabola that intersect at the point (a, b) . Find the value of $|ab|$.

[Answer: 52]

Noting that only two terms include y and rearranging terms accordingly yields $2x^3 - 8x^2 + 2xy - 8y - 6x + 24 = 0$, and factoring two terms at a time yields $2x^2(x - 4) + 2y(x - 4) - 6(x - 4) = 0$, or $(2x^2 - 6 + 2y)(x - 4) = 0$. This describes the equations $x = 4$ (a vertical line) and $y = 3 - x^2$ (a parabola). They intersect at the point $(4, -13)$, making the desired quantity $|4(-13)| = 52$.

- T-3 Point B lies on the surface (not the base) of a hemisphere with a surface area (including the base) of 675π . Point B lies directly over diameter \overline{AC} , and point D lies on \overline{AC} directly beneath B . If $AD = 2DB$, then the volume of the figure formed by rotating triangle ABC about the diameter \overline{AC} is $k\pi$. Find the value of k .

[Answer: 1440]

Because triangle ABC has a vertex on the surface of the hemisphere and has the diameter of one side, it follows that angle ABC is a right angle. Note also that since the surface area of a hemisphere is $3\pi r^2$, it follows that $r = 15$ and therefore $AC = 30$. Also $(AD)(DC) = (DB)^2$, letting $DB = x$ yields the equation $(2x)(30 - 2x) = x^2$, or $60 - 4x = x$, yielding $x = 12$. Therefore $DB = 12$, $AD = 24$, and $DC = 6$, and the figure formed by rotating the triangle about the diameter will be two cones with heights 24 and 6 that share a congruent circular base with radius 12, making the volume $\frac{1}{3}\pi(12^2)(24) + \frac{1}{3}\pi(12^2)(6) = \frac{1}{3}\pi(12^2)(30) = 1440\pi$, making the desired quantity 1440.

- T-4 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x + 105}$, find the sum of all integers a such that $f(a)$ and $g(a)$ are both integers.

[Answer: 3040]

Let $x = k^2$, and so $x + 105 = (k + p)^2 = k^2 + 2kp + p^2$. This means $105 = 2kp + p^2 = p(2k + p)$. This means p must be a factor of 105 that is less than $\frac{105}{p}$. Then the corresponding value of k will be $\frac{1}{2}\left(\frac{105}{p} - p\right)$. The values of p that work are 1, 3, 5, and 7, yielding corresponding values of k of 52, 16, 8, and 4, making the desired quantity $52^2 + 16^2 + 8^2 + 4^2 = 3040$.

- T-5 A polynomial $f(x)$ has the property that $f(0) = 2$ and for all x , $f(x + 1) - f(x) = x^3 + 8$. If the degree of $f(x)$ is n , the leading coefficient of $f(x)$ in standard form is A and the sum of the remaining coefficients of $f(x)$ is B , find $n + \frac{B}{A}$.

[Answer: 43]

Note that for a first difference to be cubic, $f(x)$ should be a quartic polynomial, so $n = 4$. Then knowing $f(x) = ax^4 + bx^3 + \dots$ and $f(x + 1) = a(x + 1)^4 + b(x + 1)^3 + \dots$, The expansion of the first two terms of $f(x + 1) - f(x) = ax^4 + 4ax^3 + 6ax^2 + 4ax + a + bx^3 + 3bx^2 + 3bx + 1 - ax^4 - bx^3$, and since $4a + b - b = 1$, we have $a = \frac{1}{4}$. Next note since $f(0) = 2$, then $f(1) - f(0) = 0^3 + 8$, so $f(1) = f(0) + 8 = 2 + 8 = 10$. Finally, note that $f(1)$ must also be the sum of the coefficients of $f(x)$. Therefore, $A = \frac{1}{4}$ and $B = 10 - \frac{1}{4} = \frac{39}{4}$, making the desired quantity $n + 4\left(\frac{39}{4}\right) = 4 + 39 = 43$.

- T-6 In how many ways can we arrange letters of “bluegreen” such that the first four letters are the letters of “blue” and the last five letters are the letters of “green”, but no two adjacent letters are the same?

[Answer: 756]

First note that there are $4! = 24$ possible arrangements of the letters of “blue” and $\frac{5!}{2} = 60$ possible arrangements of the letters of “green”, noting the e’s are interchangeable. Then

subtracting off the $4!$ arrangements of “green” where the two e’s are adjacent (treating them effectively as one letter) yields $\frac{5!}{2} - 4! = 36$ possible arrangements of the letters of “green”, making a total of $24 * 36$ arrangements to start. However, note that there are $3!$ arrangements of “blue” with e on the right-most position, and $1 * 3 * 3 * 2 * 1 = 18$ arrangements of “green” with an e in the left-most position. Therefore the number of desired arrangements would be $24 * 36 - 6 * 18 = 864 - 108 = 756$.