FAIRFIELD COUNTY MATH LEAGUE 2025-2026
Match 5
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Fractions and Exponents

1-1 Find the sum of all values n such that $\frac{2^{n^2-1}}{8^{n+1}} = 64$.
[Answer: 3]

We have $\frac{2^{n^2-1}}{(2^3)^{n+1}} = \frac{2^{n^2-1}}{2^{3n+3}} = 64 = 2^6$, so $n^2 - 1 - (3n+3) = 6$, or $n^2 - 3n + 2 = 0$. This could be solved to yield $n = 2$ or $n = 1$, making the desired quantity $2 + 1 = 3$.

1-2 The value of $\left(\frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48} + \cdots.\right)^2$ can be expressed as $\frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 10]

Note that the sum can be written as $\left(\frac{1}{6}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)\right)^2 = \left(\frac{1}{6}(1+1)\right)^2 = \frac{1}{9}$, making the desired quantity $1 + 9 = 10$.

1-3 If x and y are real numbers such that $\left(\frac{12^x}{288^y}\right)^x = 36$, find the integer value of $(64^y)^y$.
[Answer: 16]

We have $\left(\frac{(2^2*3)^x}{(2^5*3^2)^y}\right)^x = \frac{2^{2x^2}*3^{x^2}}{2^{5xy}*3^{2xy}} = 2^2 * 3^2$, yielding the system $\begin{cases} 2x^2 - 5xy = 2 \\ x^2 - 2xy = 2 \end{cases}$. Multiplying the second equation by 2 and subtracting the first equation yields $xy = 2$, which also yields $x^2 = 6$. This makes $y = \frac{2}{\sqrt{6}}$ (x could be interpreted as negative with no change to the result), and so $(64^y)^y = 64^{y^2} = 64^{\frac{2}{3}} = 16$.

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Round 2: Rational Expressions and Equations

2-1 The equation $\frac{x^2}{x-2} = \frac{k}{x-2} + 3$, with real constant k, has an extraneous solution. Solve for k.
[Answer: 4]

Multiplying both sides by $x - 2$ yields $x^2 = k + 3(x - 2)$. Since the original equation cannot have $x = 2$ as a solution, substituting this in yields $4 = k$, which is our desired result.

2-2 A 30% saline solution of unknown volume is added to 10mL of a 70% saline solution. The mixture is then diluted with an extra 1mL of pure water. The resultant mixture is a 40% saline solution. How many mL of the 30% saline solution were there initially?
[Answer: 26]

Let the unknown volume in milliliters be v. Then we have $\frac{.30v+(10)(.70)}{v+10+1} = .40$, which yields $.30v + 7.0 = .40v + 4.4$, making $.10v = 2.6$, or $v = 26$.

2-3 The graph of the function $f(x) = \frac{x+\frac{a}{x+1}}{x+5+\frac{b}{x+1}}$, where a and b are integer constants, has removable discontinuities (holes) at the points $(-1, y_1)$ and $(4, y_2)$ and a vertical asymptote at $x = k$. The quantity $|ky_1y_2|$ is $\frac{p}{q}$, where p and q are positive integers with no common factors greater than 1. Find $p + q$.
[Answer: 27]

In order for there to be a removable discontinuity, both the numerator and denominator of the original function should equal 0 when $x = 4$. This yields the equations $4 + \frac{a}{5} = 0$ and $9 + \frac{b}{5} = 0$, yielding $a = -20$ and $b = -45$. Now multiplying the numerator and denominator of the original function yields $\frac{x(x+1)-20}{(x+5)(x+1)-45} = \frac{x^2+x-20}{x^2+6x-40} = \frac{(x+5)(x-4)}{(x+10)(x-4)} = \frac{x+5}{x+10}$ for $x \neq 4$. This means $y_1 = \frac{-1+5}{-1+10} = \frac{4}{9}$ and $y_2 = \frac{4+5}{4+10} = \frac{9}{14}$. Additionally it is now clear that there is a vertical asymptote at $x = -10$. Therefore $|ky_1y_2| = \left|(-10)\left(\frac{4}{9}\right)\left(\frac{9}{14}\right)\right| = \frac{20}{7}$, making the desired quantity $20 + 7 = 27$.

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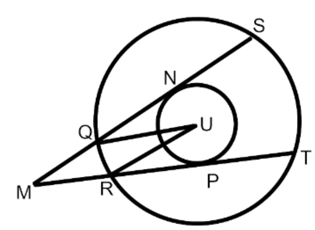
Round 3: Circles

3-1 A circle is inscribed in ΔGHI such that $\overline{GH}$ is tangent to the circle at point J, $\overline{HI}$ is tangent to the circle at point K and $\overline{GI}$ is tangent to the circle at point L. If $m\angle I = 52°$ and $m\widehat{JK} = 135°$, find $m\angle G$ in degrees.
[answer: 83]

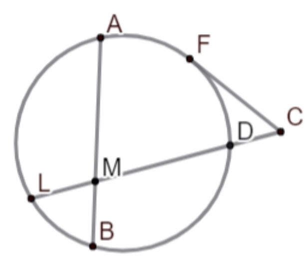
Let the center of the circle be M. Because $JHKM$ is a kite with angles K and J being right angles due to tangency with the circle, it follows that since angle M of the kite must have a measure of $135°$, angle H has a measure of $180° - 135° = 45°$. This means that $m\angle G = 180° - 52° - 45° = 83°$.

3-2 See the diagram (not drawn to scale). Both circles have center U. $\overline{MS}$ is tangent to the smaller circle at N and $\overline{MT}$ is tangent to the smaller circle at P. If $m\,\widehat{ST}$ is double the $m\,\widehat{QR}$ and $m\angle QUR = 40°$, find $m\,\widehat{NP}$ in degrees.
[answer: 160]



Because $m\,\widehat{QR} = 40°$ due to the central angle, it follows that $m\,\widehat{ST} = 2(40°) = 80°$. Therefore $m\angle M = \frac{80-40}{2} = 20°$, which means that $m\angle NUP = 180 - 20 = 160°$, which must also be the value of $m\,\widehat{NP}$.

3-3 See the diagram (not drawn to scale). Chords $\overline{AB}$ and $\overline{DL}$ intersect at point M, $\overline{FC}$ is tangent to the circle, and D lies on $\overline{CL}$. If DM and ML are solutions to the equation $x^2 - 40x + 51 = 0$, AM and MB are integers greater than 1, and $CF = 15$, find the value of $AB + CD$.
[Answer: 25]



Based on the equation, it follows that $DM + ML = 40$ and $(DM)(ML) = 51$. It follows that $(AM)(MB) = 51$, and because the lengths of the segments must be integers greater than 1, they must be 3 and 17. Next since $(FC)^2 = CD(CD + DL)$, we have $15^2 = CD(CD + 40)$, so $225 = (CD)^2 + 40(CD)$, making $CD = 5$, and thus the desired quantity is $20 + 5 = 25$.

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Round 4: Quadratic Equations & Complex Numbers

4-1 A quadratic function with real number coefficients $f(x) = x^2 + bx + c$ has a zero of $x = 3 - 2i$. Find the value of $b + c$.
[Answer: 7]

One way to solve this is to write $x - 3 = -2i$, and so $(x - 3)^2 = -4$, or $x^2 - 6x + 13 = 0$. Therefore the desired quantity is $-6 + 13 = 7$.

4-2 The quadratic equation $(1 + i)z^2 + (a + bi)z + 6 = 0$ has a root at $z = 2$. What is the value of $|ab|$?
[Answer: 10]

Substituting in $z = 2$ yields $4(1 + i) + 2(a + bi) + 6 = 0$. This means $4 + 2a + 6 = 0$, yielding $a = -5$, and $4 + 2b = 0$, yielding $b = -2$, making the desired quantity $|(-5)(-2)| = 10$.

4-3 If $z = a + bi$ is a complex number with $|z| > 0$ such that $z^2 + z \cdot \bar{z} - (14a + (b^2 + 40)i) = 0$, find the sum of all possible values of $|z|^2$.
[Answer: 214]

Because $z^2 = (a + bi)^2 = a^2 - b^2 + 2abi$ and $z \cdot \bar{z} = (a + bi)(a - bi) = a^2 + b^2$, we have $2a^2 = 14a$ and $2ab = b^2 + 40$. From the first equation we have $a = 7$ (since $a \neq 0$) and when substituted into the second equation we get $14b = b^2 + 40$, or $b^2 - 14b + 40 = 0$, which factors into $(b - 10)(b - 4) = 0$, so b can be either 4 or 10. This means the two possible values of z are $7 + 4i$ or $7 + 10i$, making the desired quantity $7^2 + 4^2 + 7^2 + 10^2 = 214$.

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Round 5: Trigonometric Equations

5-1 If x is a quadrant 1 angle where $\cot^2(x) + \sec^2(x) - \tan^2(x) = \frac{10}{7}$ such that $\sin^2(x) = \frac{a}{b}$ where a and b share no common factors, determine $b^2 - a^2$.
[Answer: 51]

Substituting $\sec^2(x) = \tan^2(x) + 1$ yields $\cot^2(x) + \tan^2(x) + 1 - \tan^2(x) = \cot^2(x) + 1 = \csc^2(x) = \frac{10}{7}$, meaning $\sin^2(x) = \frac{7}{10}$, making the desired quantity $100 - 49 = 51$.

5-2 Let x be an acute angle such that if $\sin(x) + \cos(x) = \frac{\sqrt{5}}{2}$. If $\sin^3(x) + \cos^3(x) = \frac{a\sqrt{c}}{b}$, where c is prime and a and b share no common factors. Determine $a + b + c$.
[Answer: 28]

Squaring the given equation yields $\sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x) = \frac{5}{4}$, and substituting $\sin^2(x) + \cos^2(x) = 1$ yields $\sin(x)\cos(x) = \frac{1}{8}$. Cubing the original equation yields $\sin^3(x) + 3\sin^2(x)\cos(x) + 3\sin(x)\cos^2(x) + \cos^3(x) = \frac{5\sqrt{5}}{8}$. The middle two terms can be factored as $3\sin(x)\cos(x)\,(\sin(x) + \cos(x))$, or $3\left(\frac{1}{8}\right)\left(\frac{\sqrt{5}}{2}\right) = \frac{3\sqrt{5}}{16}$. Therefore $\sin^3(x) + \cos^3(x) = \frac{5\sqrt{5}}{8} - \frac{3\sqrt{5}}{16} = \frac{7\sqrt{5}}{16}$, making the desired quantity $7 + 5 + 16 = 28$.

5-3 If θ is an angle in quadrant I such that $\frac{196}{\tan^2(2\theta)} = \sec^2(\theta) + 2\tan(\theta)$, then the sum of all possible values of $\tan(\theta)$ is $\frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 73]

Substituting $\sec^2(\theta) = \tan^2(\theta) + 1$ yields $\frac{196}{\tan^2(\theta)} = \tan^2(\theta) + 2\tan(\theta) + 1 = (\tan(\theta) + 1)^2$, or $196 = \tan^2(2\theta)(\tan(\theta) + 1)^2$. Therefore $\tan(2\theta)\,(\tan(\theta) + 1) = \pm 14$. This means $\left(\frac{2\tan(\theta)}{1-\tan^2(\theta)}\right)(\tan(\theta) + 1) = \pm 14$. Since the angle is in quadrant I, it follows that $\tan(\theta) \neq -1$, so we can divide out $1 + \tan(\theta)$ to make $\frac{2\tan(\theta)}{1-\tan(\theta)} = \pm 14$. This presents two cases in solving for $\tan(\theta)$: $\frac{2\tan(\theta)}{1-\tan(\theta)} = 14$, which yields $\tan(\theta) = \frac{14}{16} = \frac{7}{8}$, and $\frac{2\tan(\theta)}{1-\tan(\theta)} = -14$, which yields $\tan(\theta) = \frac{14}{12} = \frac{7}{6}$. This makes the sum of the values of $\tan(\theta)$ $\frac{7}{8} + \frac{7}{6} = \frac{49}{24}$, making the desired quantity $49 + 24 = 73$.

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Round 6: Sequences & Series

6-1 An arithmetic sequence has common difference d and first term a_1. If $a_{2026} = a_{20} + a_{26}$, find $\frac{a_1}{d}$.
[Answer: 1981]

Since $a_n = a_1 + (n-1)d$ for $n \geq 1$, we have $a_1 + 2025d = a_1 + 19d + a_1 + 25d = 2a_1 + 44d$, which means $a_1 = 1981d$, and so $\frac{a_1}{d} = 1981$.

6-2 The sum of an infinite geometric series is 5 and the first term is 2. If the sum of the squares of all the terms in this series can be represented as $\frac{a}{b}$, where a and b are mutually prime, what is $a - b$?
[Answer: 21]

From the given information, if the series has common ratio r, we have $\frac{2}{1-r} = 5$, which means $r = \frac{3}{5}$. The sum of the squares of all the terms is equivalent to the sum of an infinite geometric series with first term 4 and common ratio $\frac{9}{25}$, making the sum $\frac{4}{1-\frac{9}{25}} = \frac{4}{\frac{16}{25}} = \frac{25}{4}$, making the desired quantity $25 - 4 = 21$.

6-3 A sequence of strictly increasing integers $a_1, a_2, \ldots$ follows the rule that $a_n = 2a_{n-1} + 2026$. What is the smallest possible value of a_{12}?
[Answer: 22]

Because the sequence is strictly increasing, we have $a_n > a_{n-1}$, so $2a_{n-1} + 2026 > a_{n-1}$, making $a_{n-1} > -2026$. This means that all terms in the sequence must be greater than -2026, and since the terms are integers, the smallest possible value of a_1 is -2025. From this, we see $a_2 = -2024$, $a_3 = -2022$, etcetera, and so the pattern yields $a_n = -2026 + 2^{n-1}$, making $a_{12} = -2026 + 2^{11} = -2026 + 2048 = 22$.

FAIRFIELD COUNTY MATH LEAGUE 2025-2026
Match 5
Team Round

Please write your answers on the answer sheet provided.

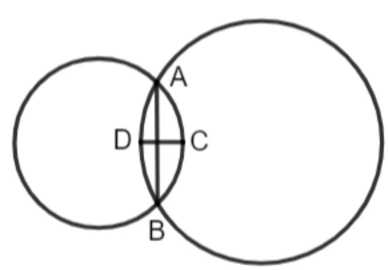
T-1 Let M be the set of all positive integers with 36 factors (including one and itself) that have both 2 and 5 as their only prime factors. Let N be the set of all positive integers with 12 factors (including one and itself) that have both 2 and 5 as their only prime factors. Let S be the set of all possible integer quotients of a number from set M and a number from set N. The product of the greatest and least elements in S is $a^b * 10^c$, where $a, b,$ and c are positive integers and a is prime. Find $a + b + c$.
[Answer: 17]

Instead of finding all the elements of S, we can identify the candidates for the greatest and least elements. The largest element of M is $2 * 5^{17}$; the largest element of S should maximize the exponent of 5 and the smallest should minimize the exponent of 5. One element of N is $2 * 5^5$, and therefore the largest element in S is 5^{12}. Next note that we can reduce the exponent of 5 to 0 while minimizing the exponent of 2 by taking the element $2^5 * 5^5$ in M and dividing by $2 * 5^5$ in N to get 2^4, the smallest element of S. The product of these two quotients is $2^4 * 5^{12} = 5^8 * 10^4$, making our desired quantity $5 + 8 + 4 = 17$.

T-2 Rapido the rabbit races Vite the viper down a straight track. They both keep a constant speed, except when Rapido injures his foot halfway down the track, reducing his speed by half. This allows Vite to pass Rapido ¾ down the track. After Vite crosses the finish line, it takes Rapido another 20 seconds to run the last 30 feet of track. What is the length of the track in feet?
[Answer: 360]

Let Rapido's starting speed in feet per second be r and let Vite's speed in feet per second be v. The length of the track in feet will be D. The time it takes Rapido to run have the track at speed r and another quarter at speed $\frac{1}{2}r$ is the same time it takes Vite to run $\frac{3}{4}D$ at speed v, so $\frac{\frac{1}{2}D}{r} + \frac{\frac{1}{4}D}{\frac{1}{2}r} = \frac{\frac{3}{4}D}{v}$, which can be solved to yield $v = \frac{3}{4}r$. We also know that Rapido's speed in the second half, or $\frac{r}{2}$, is $\frac{3}{2}$ feet per second, meaning $r = 3$ feet per second and $v = \frac{9}{4}$ feet per second. Finally, after running $\frac{1}{4}D$ feet, Vite was 30 feet ahead of Rapido, so $\frac{\frac{1}{4}D}{\frac{9}{4}} = \frac{\frac{1}{4}D-30}{\frac{3}{2}}$, or $\frac{3}{8}D = \frac{9}{16}D - \frac{135}{2}$, which can be solved to yield $D = 360$.

T-3 See the diagram (not drawn to scale). Circles P with radius 6 and Q with radius 10 intersect each other at points A and B, with point C on P and point D on Q such that $\overline{CD}$ is a perpendicular bisector of $\overline{AB}$. If $\overline{AB}$ cuts $\overline{CD}$ into a 2:1 ratio, then $(AB)^2 = \frac{m}{n}$ where m and n are positive integers with no common factors greater than 1. Find $m + n$.
[Answer: 905]



Let the midpoint of $\overline{AB}$ be E, which means $DE = x$, $EC = 2x$, and $AE = EB = y$. Using intersecting chords, we have $y^2 = 2x(12 - 2x) = x(20 - x)$. This means $3x^2 = 4x$, or $x = \frac{4}{3}$. Since $(AB)^2 = (2y)^2 = 4y^2$, we have $4y^2 = 4x(20 - x) = 4\left(\frac{4}{3}\right)\left(20 - \frac{4}{3}\right) = 4\left(\frac{4}{3}\right)\left(\frac{56}{3}\right) = \frac{896}{9}$, making the desired quantity $896 + 9 = 905$.

T-4 There exists a complex number $z = a + bi$ where a and b are real numbers such that $a = 2b$ and $|(1 + i)(1 + z + z^2 + z^3 + \cdots)| = 3$. The product $ab = \frac{p}{q}$ where p and q are positive integers with no factors greater than 1. Find $p + q$.
[Answer: 11]

Since the series converges, it follows that $\left|\frac{1+i}{1-z}\right| = 3$, and therefore $|1 - z| = \frac{\sqrt{2}}{3}$, and $|1 - z|^2 = \frac{2}{9}$. This means $(1 - 2b)^2 + b^2 = \frac{2}{9}$, or $5b^2 - 4b + 1 = \frac{2}{9}$, or $45b^2 - 36b + 7 = 0$. This is factorable into $(3b - 1)(15b - 7) = 0$, yielding $b = \frac{1}{3}$ or $b = \frac{7}{15}$. The latter is extraneous, since $\left|\frac{14}{15} + \frac{7}{15}i\right| > 1$, so $b = \frac{1}{3}$ and $a = \frac{2}{3}$, and so $ab = \frac{2}{9}$, making the desired quantity $2 + 9 = 11$.

T-5 Consider right triangle AFL with right angle L. Points C and M lie on $\overline{FL}$ such that $\angle FAC \cong \angle CAM \cong \angle MAL$. If $AL = 12$ and $ML = 4$, then $FL = \frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 55]

Let $m\angle MAL = \theta$, and so $\tan(\theta) = \frac{4}{12} = \frac{1}{3}$. $m\angle CAL = 2\theta$, and $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)} = \frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{3}{4}$. $m\angle FAL = 3\theta = \tan(2\theta + \theta) = \frac{\tan(2\theta)+\tan(\theta)}{1-\tan(2\theta)\tan(\theta)} = \frac{\frac{3}{4}+\frac{1}{3}}{1-\left(\frac{3}{4}\right)\left(\frac{1}{3}\right)} = \frac{\frac{13}{12}}{1-\frac{1}{4}} = \frac{13}{12} * \frac{4}{3} = \frac{13}{9}$. Since $FL = 12\tan(3\theta)$, we have $FL = 12\left(\frac{13}{9}\right) = \frac{52}{3}$, making the desired quantity $52 + 3 = 55$.

T-6 The sum of the first two terms of geometric series S_1 (first term 2025, ratio r) is 40% of the non-zero sum of the first eight terms. What is the sum of the first 4 terms of geometric series S_2 (first term 2026, ratio r^2)?
[Answer: 5065]

From the first statement, we have $\frac{2025(1-r^2)}{1-r} = \frac{2}{5}\left(\frac{2025(1-r^8)}{1-r}\right)$, or $\frac{1-r^2}{1-r^8} = \frac{2}{5}$, or $\frac{1}{(1+r^2)(1+r^4)} = \frac{2}{5}$. (Note:
$r = 1$ and $r = -1$ are both extraneous solutions, the latter because it would make the sum zero and the problem states that the sum is not zero.) This means $1 + r^2 + r^4 + r^6 = \frac{5}{2}$. The sum of the first 4 terms of S_2 is $2026(1 + r^2 + r^4 + r^6) = 2026\left(\frac{5}{2}\right) = 5065$, which is our desired quantity.