FAIRFIELD COUNTY MATH LEAGUE 2025–2026
Match 4
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Basic Statistics

1-1 A set of 6 positive integers has a mean of 12, a unique mode of 8, and a median of 10. What is the largest possible value of the largest number in the set?
[Answer: 30]

Note that the sum of the integers must be 72, there must be two 8's, and the average of the third and fourth elements must be 10. To minimize the lower five values, let them equal $1, 8, 8, 12,$ and 13 (as 8 can be the only repeated element). Therefore the desired quantity is $72 - 13 - 12 - 8 - 8 - 1 = 30$.

1-2 Alvin is playing 15 games as part of a bowling season. His average in games 7-10 is 25 greater than his average in games 1-6. His average in games 11-15 is $33\frac{1}{3}\%$ greater than his average in games 7-10, making his season average 30 points greater than his average in games 1-6. What was his season average?
[Answer: 140]

Let x be the average of the first 6 games, with a total score of $6x$. The average of the next four games is $x + 25$, with a total score of $4x + 100$. Since his season average is $x + 30$, his total score for the season must be $15x + 450$, which makes the total score for the last five games $15x + 450 - (4x + 100) - (6x) = 5x + 350$, and therefore the average of the last 5 games is $x + 70$. Setting up $x + 70 = \frac{4}{3}(x + 25)$ yields $3x + 210 = 4x + 100$ and therefore $x = 110$, making his season average $110 + 30 = 140$.

1-3 The geometric mean of a set of n numbers is defined as the n^{th} root of the product of the numbers. Two numbers a and b have a geometric mean of 30. Increasing both a and b by a number k increases their arithmetic mean by 6 and increases their geometric mean by 12. Find $a + b$.
[Answer: 138]

We know that $\sqrt{ab} = 30$ and $\sqrt{(a + k)(b + k)} = 42$, and since $\frac{a+k+b+k}{2} = \frac{a+b}{2} + 6$, we know $k = 6$. Expanding the left side of the second equation and squaring both sides yields $ab + 6(a + b) + 36 = 1764$, and substituting $ab = 900$ we have $6(a + b) + 936 = 1764$, yielding $6(a + b) = 828$ and therefore $a + b = 138$.

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Round 2: Quadratic Equations

2-1 The quadratic $y = ax^2 + bx + c$ where $a, b,$ and c are rational numbers has a zero of $-4 + 3i$ and contains the point (1, 17). If a, *b,* and *c* are real numbers, find the product abc.
[Answer: 25]

Note that the other zero must be $-4 - 3i$. The sum of these zeros is -8 and the product is 25. This makes the desired quadratic $y = a(x^2 + 8x + 25)$. Substituting the point $(1,17)$ yields $17 = a(34)$, so $a = \frac{1}{2}$. Therefore $a = \frac{1}{2}$, $b = 4$, and $c = \frac{25}{2}$, making the desired quantity $\left(\frac{1}{2}\right)(4)\left(\frac{25}{2}\right) = 25$.

2-2 Consider the functions $f(x) = 2mx^2 + 2mx + 5$ and $g(x) = mx^2 + 3x + 3$, where m is a real number. If $f(x)$ and $g(x)$ intersect at one point, then k is the sum of all possible values of m. Find $24k$.
[Answer: 120]

Setting $2mx^2 + 2mx + 5 = mx^2 + 3x + 3$, we have $mx^2 + (2m - 3)x + 2 = 0$. For there to be only one solution, require $(2m - 3)^2 - 4(2)m = 0$, or $4m^2 - 20m + 9 = 0$. This makes the sum of all possible values of $m = \frac{20}{4} = 5$, making the desired quantity $(24)(5) = 120$.

2-3 What is the smallest positive integer a such that the equation $x^2 - ax + a + 2025 = 0$ has integer solutions?
[Answer: 1017]

Note we are looking for two numbers, p and q, such that $pq = p + q + 2025$. Arbitrarily isolating p yields $p = \frac{q+2025}{q-1} = \frac{q-1+2026}{q-1} = 1 + \frac{2026}{q-1}$. Therefore, for p to be an integer, $q - 1$ must be a factor of 2026. The factors of 2026 are $1, 2, 1013,$ and 2026, so the sum is minimized when $q = 3$ and $p = 1014$ (or vice-versa), making the desired quantity $3 + 1014 = 1017$.

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Round 3: Similarity

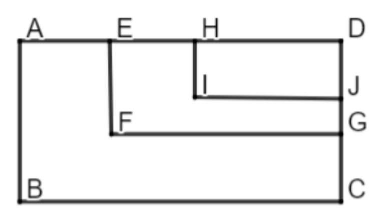
3-1 Jerry stands near a 32 foot tall tree. He is 6 feet tall. His shadow is 42 inches long. The tips of his shadow and that of the tree are at the same point. Exactly how close (in inches) to the tree does he stand?
[Answer: 182]

Setting up a proportion in feet, we have $\frac{32}{x} = \frac{6}{3.5}$, which yields $x = \frac{(3.5)(32)}{6} = \frac{112}{6} = \frac{224}{12}$, making the tree's shadow 224 inches long. Jerry's distance to the tree is that length minus the length of his shadow, or $224 - 42 = 182$.

3-2 There exists an isosceles triangle such that the height is 8 and the numerical values of its perimeter and area are the same. There exists a similar triangle whose area is twice the area of the first triangle. Find the perimeter of the second triangle.
[Answer: 32]

Let half of the base of the isosceles triangle be x. Then the area is $8x$ and the perimeter is $2x + 2\sqrt{x^2 + 64}$. Setting them equal yields $8x = 2x + 2\sqrt{x^2 + 64}$, so $3x = \sqrt{x^2 + 64}$. Squaring both sides yields $9x^2 = x^2 + 64$, or $8x^2 = 64$, leading to $x^2 = 8$ and therefore $x = 2\sqrt{2}$, and the perimeter of the triangle is $8(2\sqrt{2}) = 16\sqrt{2}$. Since the larger triangle has an area that is twice that of the smaller, its perimeter will be greater than that of the smaller by a factor of $\sqrt{2}$, making the desired quantity $(16\sqrt{2})(\sqrt{2}) = 32$.

3-3 See the diagram (not drawn to scale). A rectangular pen $ABCD$ is to be constructed such that there are similar rectangular pens, $EFGD$ and $HIJD$, and $\frac{\text{area } ABCD}{\text{area } EFGD} = \frac{\text{area } EFGD}{\text{area } HIJD}$. If the amount of material used to construct the required additional sides for $EFGD$ and $HIJD$ is equal to half the material required to construct $ABCD$, then $\frac{\text{area } HIJD}{\text{area } ABCD} = \frac{u - v\sqrt{w}}{z}$, where u, v, and w are positive integers and u and w have no common factors greater than 1. Find $u + v + w + z$.
[Answer: 17]



Let $BC = x$ and k be the value of $\frac{FG}{BC}$, so $FG = kx$, and consequently $IJ = k^2x$. From the problem, we have $IJ + FG = BC$, so $k^2x + kx = x$, which yields $k^2 + k - 1 = 0$. Using the quadratic formula yields $k = \frac{-1 \pm \sqrt{5}}{2}$, but since $0 < k < 1$, we know $k = \frac{\sqrt{5}-1}{2}$. Because $\frac{\text{area } HIJD}{\text{area } DEFG} = \frac{\text{area } DEFG}{\text{area } ABCD} = k^2$, we have $\frac{\text{area } HIJD}{\text{area } ABCD} = k^4$, and since $k^2 = \left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}$, then $k^4 = \left(\frac{3-\sqrt{5}}{2}\right)^2 = \frac{14-6\sqrt{5}}{4} = \frac{7-3\sqrt{5}}{2}$, making the desired quantity $7 + 3 + 5 + 2 = 17$.

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Round 4: Variation

4-1 The value of y varies directly with the square of x, and $y = 4$ when $x = 4$. What is the positive value of x when $y = 100$?
[Answer: 20]

Setting up $\frac{4}{4^2} = \frac{100}{x^2}$ yields $x^2 = 400$, which means $x = 20$.

4-2 The value of a varies directly with b. Using the same positive constant of proportionality, the value of b varies inversely with c. There exists an integer x such that if $c = x$ then $a = x + 5$. What is the constant of proportionality?
[Answer: 6]

Setting up $a = kb$ and $b = \frac{k}{c}$, we have $a = \frac{k^2}{c}$, and so $ac = x(x+5) = k^2$, or $x^2 + 5x - k^2 = 0$. In order for x to be an integer, there must be an integer m such that $5^2 + 4k^2 = m^2$, or $m^2 - 4k^2 = 25$. This means $(m-2k)(m+2k) = 25$, which means that $(m-2k, m+2k)$ either equals $(1,25)$ or $(5,5)$. The latter can be excluded since it means $k = 0$. Therefore $m + 2k = 25$ and $m - 2k = 1$, meaning $m = 13$ and $k = 6$, which is our desired quantity.

4-3 Three positive values x, y, and z are related such that z varies with directly with x^3 and inversely with y to a power of $n > 0$. However, x and y are not independent; y varies directly with x to the power of $2n + 1$, which consequently allows z to be expressed (without y) as varying directly with x^2. If $z = 45$ when $y = 50$ and $x = 30$, then when $z = 54$, $\frac{x}{y} = \frac{\sqrt{a}}{b}$ where a and b are positive integers and a contains no perfect square factors greater than 1. Find $a + b$.
[Answer: 40]

From the first relationship, we have $z = \frac{k_1 x^3}{y^n}$, and from the second relationship, we have $y = k_2 x^{2n+1}$. Combining the two relationships yields $\frac{k_1 x^3}{(k_2 x^{2n+1})^n} = k_3 x^2$, which means that $(x^{2n+1})^n = x$, which yields $2n^2 + n - 1 = 0$. This is solvable by factoring to yield $n = -1$ and $n = \frac{1}{2}$, the former of which is extraneous because $n > 0$. This means that

$z \propto x^2$ (from the problem) and $y \propto x^{2\left(\frac{1}{2}\right)+1} = x^2$, which also means $z \propto y$. Since $\frac{45}{30^2} = \frac{54}{x^2}$, then $x^2 = \frac{6}{5}(30)^2$, so $x = 6\sqrt{30}$, and since $\frac{45}{50} = \frac{54}{y}$, we have $y = 60$, and so $\frac{x}{y} = \frac{6\sqrt{30}}{60} = \frac{\sqrt{30}}{10}$, making the desired quantity $30 + 10 = 40$.

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Round 5: Trig Expressions & DeMoivre's Theorem

5-1 Let $z = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$. If $z^4 = a + \frac{c\sqrt{d}}{f}i$ where c and f are positive integers no common factors greater than 1 and d is a positive integer with no perfect square factors greater than 1, find $af + cd$.
[Answer: 2]

We have $z^4 = \cos\left(\frac{4\pi}{6}\right) + i\sin\left(\frac{4\pi}{6}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, making the desired quantity $\left(-\frac{1}{2}\right)(2) + (1)(3) = 2$.

5-2 Let A and B be angles such that $\tan(A) = -1$ and $\tan(B) = 2$ with $\frac{\pi}{2} < A < \pi$ and $\pi < B < \frac{3\pi}{2}$. Then $\sin(2A) + \cos(2B) + \frac{c}{g} = 0$, where c and g are positive integers with no perfect square factors greater than 1. Find cg.
[Answer: 40]

Note that $\sin(A) = \frac{\sqrt{2}}{2} = -\cos(A)$, and $\cos(B) = -\frac{1}{\sqrt{5}}$. Therefore $\sin(2A) + \cos(2B) = 2\sin(A)\cos(A) + 2\cos^2(B) - 1 = 2\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + 2\left(-\frac{1}{\sqrt{5}}\right)^2 - 1 = -1 - \frac{3}{5} = -\frac{8}{5}$, which means $\frac{c}{g} = \frac{8}{5}$, making the desired quantity $8 * 5 = 40$.

5-3 If θ is an angle in quadrant I such that $\tan\left(2\theta + \frac{\pi}{4}\right) = 5$, then $\theta = \arctan\left(\frac{\sqrt{a}-b}{c}\right)$, where $a, b,$ and c are positive integers and a has no perfect square factors greater than 1. Find $a + b + c$.
[Answer: 18]

Setting up $\tan\left(2\theta + \frac{\pi}{4}\right) = \frac{\tan(2\theta)+\tan\left(\frac{\pi}{4}\right)}{1-\tan(2\theta)\tan\left(\frac{\pi}{4}\right)} = \frac{\tan(2\theta)+1}{1-\tan(2\theta)} = 5$, we have $\tan(2\theta) + 1 = 5 - 5\tan(2\theta)$, or $\tan(2\theta) = \frac{2}{3}$. Then setting up $\frac{2\tan(\theta)}{1-\tan^2(\theta)} = \frac{2}{3}$, we have $6\tan(\theta) = 2 - 2\tan^2(\theta)$, or $\tan^2(\theta) + 3\tan(\theta) - 1 = 0$. The positive value of $\tan(\theta)$, which would yield θ in quadrant I, is therefore $\tan(\theta) = \frac{-3+\sqrt{13}}{2} = \frac{\sqrt{13}-3}{2}$, making our desired quantity $13 + 3 + 2 = 18$.

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Round 6: Conic Sections

6-1 Points $A(2,3)$ and $B(7,8)$ are on a circle with center (h,k). Find $h+k$.
[Answer: 10]

Method 1: Equation
Setting up $(2-h)^2+(3-k)^2=(7-h)^2+(8-k)^2$, we have $13-4h-6k+h^2+k^2=113-14h-16k+h^2+k^2$, or $10(h+k)=100$, yielding $h+k=10$.

Method 2: Coordinate Geometry
The center of the circle must lie on the perpendicular bisector of the chord $\overline{AB}$. Note that the slope of the line between A and B is 1, and the midpoint of A and B is $\left(\frac{9}{2},\frac{11}{2}\right)$. The equation of the perpendicular bisector is therefore $y=-\left(x-\frac{9}{2}\right)+\frac{11}{2}$, or $y=-x+10$, which means $x+y=10$, our desired quantity.

6-2 Points $A(x_1,y_1)$ and $B(x_2,y_2)$ are both on the parabola with vertex $(0,0)$ and focus $(1,0)$. If $x_1x_2+y_1y_2=12$, find the maximum value of x_1x_2.
[Answer: 36]

With a vertex of $(0,0)$, the equation is $x=\frac{1}{4p}y^2$, and since $p=1$, we have an equation of $x=\frac{1}{4}y^2$ or $y^2=4x$, which means $y_1^2=4x_1$ and $y_2^2=4x_2$, and consequently $y_1^2y_2^2=16x_1x_2$. Squaring both sides of the equation $y_1y_2=12-x_1x_2$ yields $y_1^2y_2^2=x_1^2x_2^2-24x_1x_2+144$, and substituting yields $16x_1x_2=x_1^2x_2^2-24x_1x_2+144$, or $x_1^2x_2^2-40x_1x_2+144=0$. This means $x_1x_2=36$ or $x_1x_2=4$, making 36 our desired quantity.

6-3 For given values of a and b, consider an ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and a hyperbola with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The ellipse has a focus $(0,6)$ and an area, given by πab, of $k\pi$. The hyperbola has a focus $\left(2\sqrt{15},0\right)$ and an asymptote that passes through the point $(10,p)$. Find $k+p^2$.
[Answer: 424]

From the ellipse, we know that $b^2>a^2$ since the focus lies on the vertical axis, and therefore $b^2-a^2=36$. From the hyperbola, we know $b^2+a^2=60$. Solving this system yields $b = 4\sqrt{3}$ and $a = 2\sqrt{3}$. The area of the ellipse is therefore $\pi(2\sqrt{3})(4\sqrt{3}) = 24\pi$, so $k = 24$. The asymptotes of the hyperbola are therefore $y = \pm\frac{b}{a}x = \pm 2x$, which means they contain the point $(10, \pm 20)$, so $p^2 = 400$, making the desired quantity $24 + 400 = 424$.

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Match 4
Team Round

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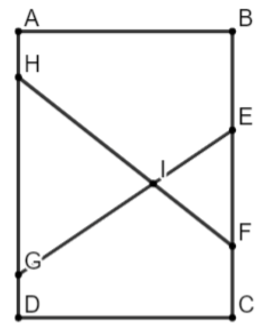
1. Sets A and B both consist of 5 distinct positive integers. Q_1, Q_2 and Q_3 of set A are equal to Q_0, Q_1, and Q_3 of set B, respectively. Set B has the property that both its mean and median are equal to 10, which is also the amount by which the range of set A is larger than the range of set B. Find the number of possible sums of the elements of A.
[Answer: 13]

We will let x_0, x_1, x_2, x_3 and x_4 represent the quartiles of set A. Because the sum of the elements of set B is 50, we know the largest value of set B is $50 - x_3 - 10 - x_2 - x_1 = 40 - x_3 - x_2 - x_1$, and therefore the range of set B is $(40 - x_3 - x_2 - x_1) - x_1 = 40 - x_3 - x_2 - 2x_1$. The range of set A is therefore $x_4 - x_0 = 10 + 40 - x_3 - x_2 - 2x_1$. This means that $50 - x_1 + 2x_0 = x_0 + x_1 + x_2 + x_3 + x_4$, which is our desired quantity. The sum of the elements therefore can be computed only from x_0 and x_1. Note that since $x_0 < x_1 < x_2 < 10$, the largest possible value of x_1 is 8, the smallest value of x_0 is 1, and the largest value of x_0 is 7. Therefore the largest possible value of the sum of the elements is $50 - 8 + 2(7) = 56$ and the smallest possible value is $50 - 8 + 2(1) = 44$, and this can take any integer value in between, making a total of 13 possible sums.

2. A particular quadratic equation $ax^2 - bx + c = 0$ has the property that $a, b,$ and c are positive prime integers less than 100 and the solutions to the equation are rational numbers greater than or equal to 1. Find the largest possible value of $a + b + c$.
[Answer: 146]

First note that $a < c$ (since the solutions must be greater than 1). For the solutions to be rational, b must be the sum of two numbers whose product is ac. Because a and c are prime, these two numbers must be either 1 and ac or a and c. However, we cannot include 1 because then a solution would be $\frac{1}{a}$, which is less than 1. Therefore b must be the sum of two prime numbers and must also be prime. This also means that $a = 2$, since b must be odd. The largest prime number that is less than 100 that is also the sum of 2 and another prime number is 73, so $a = 2$, $b = 73$, and $c = 71$, making the desired quantity $2 + 73 + 71 = 146$.

3. See the diagram (not drawn to scale). For rectangle $ABCD$, $CF = 2AH$, $BE = 3DG$, $AB = EF$, the area of triangle EFI is 50 square units, and the area of triangle GHI is 450 square units. If AH and DG are both positive integers, find the largest possible value of the area of $ABCD$. [Answer: 1980]



Let $AH = x$ and $DG = y$. Therefore $BE = 2x$ and $CF = 3y$. Moreover, let $EF = z$. Since the area of the triangles is in a ratio of $9:1$, their side lengths are in a ratio of $3:1$, making $GH = 3z$. Becahse $AD = BC$, we can set up $x + y + 3z = 2x + 3y + z$, which yields $z = y + \frac{1}{2}x$, which means $AB = y + \frac{1}{2}x$. Next, the height of triangle EFI must be $\frac{1}{4}\left(y + \frac{1}{2}x\right)$, so $50 = \frac{1}{2}\left(\frac{1}{4}\left(y + \frac{1}{2}x\right)^2\right)$, or $\left(y + \frac{1}{2}x\right)^2 = 400$, so $y + \frac{1}{2}x = 20$. The area of $ABCD$ is $\left(y + \frac{1}{2}x\right)\left(4y + \frac{5}{2}x\right)$, and substituting $y = 20 - \frac{1}{2}x$, this becomes $(20)\left(80 + \frac{1}{2}x\right)$. This is maximized when $x = 38$ and $y = 2$ (to be two integers that fit the equation $y + \frac{1}{2}x = 20$), making the area $(20)(99) = 1980$.

4. The variable w varies jointly as x to the power of a and y to the power of b, and inversely as z to the power of c, where $a, b,$ and c are positive integers. The units of x are oingos, the units of y are oingo-boingos, the units of z are boingos, and the units of w are oingos cubed per boingo. If $w = 100$ when $x = 40$, $y = 20$, and $z = 36$, find the sum of all possible values of w when $x = 80, y = 120$, and $z = 72$. Assume the proportionality constant is unitless. [Answer: 1500]

From the problem, we have $w = \frac{kx^a y^b}{z^c}$. From the units we have $a + b = 3$ (for "oingos cubed") and $b - c = -1$ (for "per boingo"). This gives two possible ordered triples for positive integers $a, b,$ and c: $(1,2,3)$ and $(2,1,2)$. For the first ordered triple we have $w = \frac{100(2^1)(6^2)}{2^3} = 900$, and for the second ordered triple we have $w = \frac{100(2^2)(6^1)}{2^2} = 600$, making the desired quantity $900 + 600 = 1500$.

5. A complex number z has one complex n^{th} root with an argument of $112°$ and another complex n^{th} root with an argument of $172°$. Given the maximum possible value of $n \le 50$, find the argument of z expressed in degrees between 0 and 360. [Answer: 336]

Because the two roots are $60°$ apart, it follows that a multiple of $\frac{360}{n}$ must be 60. Therefore $k\left(\frac{360}{n}\right) = 60 \rightarrow n = 6k$, so n must be a multiple of 6. The largest possible $n \le 50$ therefore is $n = 48$. One way to get the argument of z is to compute $(112)(48)$ and then find the coterminal argument between 0 and 360 degrees. Since $(112)(48) = 5376$, we have $5376 - 14(360) = 336$, which is our desired quantity.

6. Consider an ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ that contains the point $(2,4)$ and whose area, given by πab, is 36π. If we define S as the sum of all possible positive values of a, find S^2.
[Answer: 117]

We have $\frac{4}{a^2}+\frac{16}{b^2}=1$, or $4(4a^2+b^2)=a^2b^2$. Since $ab=36$, we have $4(4a^2+b^2)=1296$, or $4a^2+b^2=324$. Adding $4ab$ to both sides yields $4a^2+4ab+b^2=324+4(36)=468$. This means $(2a+b)^2=468$, so $2a+b=6\sqrt{13}$, so $b=6\sqrt{13}-2a$. Substituting into $ab=36$ yields $a(6\sqrt{13}-2a)=36$, so $2a^2-6\sqrt{13}a+36=0$, or $a^2-3\sqrt{13}a+18=0$. This means that the sum of the possible values of a (which would both be positive) is $3\sqrt{13}$, so the desired quantity is $\left(3\sqrt{13}\right)^2=117$.