FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 6
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Lines and Angles

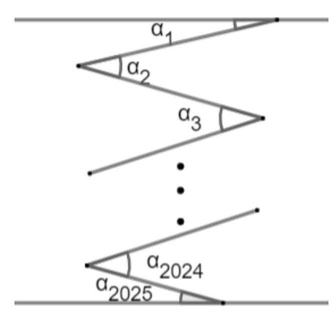
1-1 If acute angles A and B are complementary and $m\angle A = a°$ and $m\angle B = b°$, where $a < b$ and a and b are both prime integers, find the sum of the smallest and largest possible values of a.
[Answer: 50]

The smallest value of a that works is $a = 7$ since the complement $b = 83$ is prime. We also know $a \leq 45$, and so the largest value of a that works is 43 since 47 is also prime. Therefore the desired quantity is $7 + 43 = 50$.

1-2 Consider isosceles triangle ABC, with $AB = AC$ and point D on $\overline{BC}$. If $m\angle BAD = m\angle DAC + 40°$, find $m\angle ADC$ in degrees.
[Answer: 110]

Let $m\angle DAC = x$ and $m\angle ADC = y$. Therefore $m\angle BAD = x + 40$ and $m\angle ADB = y - 40$. Therefore $y + y - 40 = 180$, which can be solved to yield $y = 110$.

1-3 See the diagram (not drawn to scale). Between the parallel lines are 2025 angles labeled $\angle\alpha_1$ through $\angle\alpha_{2025}$. $m\angle\alpha_1 = m\angle\alpha_{2025} = a°$ and the measures of angles $\angle\alpha_1$ through $\angle\alpha_{2024}$ form an increasing arithmetic sequence. If the sum of the measures of all 2025 angles in degrees is ka, Find the value of k.
[Answer: 4048]



If the first angle has measure $a°$, the second angle is made up of an angle with measure $a°$ and a second with measure $k°$. The next angle is made up of an angle with measure $k°$ and a second of measure $(a + k)°$. This allows for the alternate interior angles to be congruent and makes the arithmetic sequence work. By this logic, the very last angle will have measure $\left(\frac{2025-1}{2}\right)k°$, or $(1012k)°$. This means $a = 1012k$. The total sum of all angles is $a + \sum_{n=0}^{2023}(a + nk) = 2025a + (2023)(1012)k = 2025a + 2023a = 4048a$, making the desired quantity 4048.

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Round 2: Literal Equations

2-1 There is a constant value of A such that for all x and y, $y = \frac{Ax+2}{x+5}$ and $x = \frac{2-5y}{y-4}$ contain the same ordered pairs (x, y). If one of the ordered pairs is (A, B), find the value of B.
[Answer: 2]

Solving for y in the equation where x is isolated yields $y = \frac{4x+2}{x+5}$, making $A = 4$. Substituting $x = 4$ in the first equation yields $y = \frac{4(4)+2}{4+5} = \frac{18}{9} = 2$, which is the desired quantity.

2-2 Find the sum of all values of a such that $(a, 2025)$ is a solution to $9x^2 - 6xy + y^2 = 2025$.
[Answer: 1350]

Factoring the left side yields$(3x - y)^2 = 2025$, which means $3x - y = 45$ and $3x - y = -45$. Substituting in $y = 2025$ into $3x - y = 45$ yields $x = 690$, and substituting $y = 2025$ into $3x - y = -45$ yields $x = 660$, making the desired quantity $690 + 660 = 1350$.

2-3 If m and n are positive integers such that $\frac{10}{m} + \frac{45}{n} = \frac{1}{45}$, what is the sum of the smallest possible value of n and the smallest possible value of m?
[Answer: 2477]

Solving the equation for m yields $m = \frac{450n}{n-2025}$ and solving the equation for n yields $n = \frac{2025m}{m-450}$. This means that the smallest possible value of n that produces a positive integer value of m must be $n = 2026$ and the smallest possible value of m that produces a positive integer for n must be $m = 451$, making the desired quantity $2026 + 451 = 2477$.

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Round 3: Solids & Volume

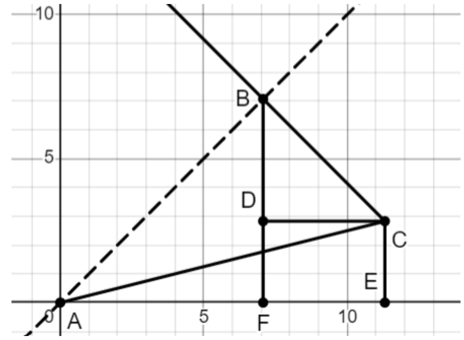
3-1 A hemisphere with a circular base and a sphere have the same radius. If the hemisphere's surface area is 10π square inches less than the sphere's surface area, then the sphere's surface area is $k\pi$ square inches. What is the value of k?
[Answer: 40]

The surface area of a hemisphere of radius r is $2\pi r^2 + \pi r^2 = 3\pi r^2$, and the surface area of a sphere with radius r is $4\pi r^2$. Therefore the difference in the surface areas is $4\pi r^2 - 3\pi r^2 = \pi r^2 = 10\pi$, so $r^2 = 10$ and the surface area of the sphere is 40π.

3-2 Mr. Bearse takes a piece of oaktag that is twice as long as it is wide. He cuts a one-inch square out of each corner and then folds the rectangular pieces upward along the dotted lines as shown to make a rectangular prism with no top face. If the resulting box has an exterior surface area of 46 square inches, find the volume of the box in cubic inches.
[Answer: 24]

Set the original width of the oaktag as x and the length as $2x$. The exterior surface area therefore has to be $2x(x) - 4(1)(1)$, yielding $2x^2 - 4 = 46$. Solving this yields $x = 5$, making the original dimensions of the oaktag 5 inches by 10 inches. The volume of the resulting box is therefore $(10-2)(5-2)(1) = 8 * 3 = 24$ cubic inches.

3-3 The segment of the line $y = kx$, with $0 < k < 1$, that has domain $0 \le x \le a$ is rotated about the line $y = x$ to produce a cone with height 10 and volume 120π. If the same segment of $y = kx$ were rotated about the x-axis, the resulting cone would have a lateral surface area of $p\sqrt{q}\pi$ where p and q are positive integers and q has no perfect square factors greater than 1. Find $p + q$.
[Answer: 25]



See the diagram. $\overline{AB}$ will be the height of the cone and has length 10, which means that $AF = FB = 5\sqrt{2}$. Additionally, we can set $120\pi = \frac{1}{3}\pi r^2(10)$ to find that $r = 6$, which means $BC = 6$. This also means $DB = DC = FE = 3\sqrt{2}$. Since $FD + DB = 5\sqrt{2}$, we know $FD = EC = 2\sqrt{2}$ and $AE = AF + FE = 8\sqrt{2}$ (incidentally, this makes $k = \frac{1}{4}$). The lateral surface area of the cone formed by rotating $\overline{AC}$ about the x-axis is $\pi(EC)(AC) = \pi(2\sqrt{2})(\sqrt{(8\sqrt{2})^2 + (2\sqrt{2})^2} = \pi(2\sqrt{2})(\sqrt{136}) = 8\sqrt{17}\pi$, making the desired quantity $8 + 17 = 25$.

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Round 4: Radical Expressions and Equations

4-1 Solve for x: $\sqrt{x-1} = \frac{6}{\sqrt{x+4}}$
[Answer: 5]

Squaring both sides yields $x - 1 = \frac{36}{x+4}$, or $x^2 + 3x - 40 = 0$, which factors into $(x+8)(x-5) = 0$. However the solution $x = -8$ is extraneous, making $x = 5$ the desired quantity.

4-2 Find the sum of all values of x that satisfy the equation $\sqrt[3]{x^2 - 4x + 4} - \sqrt[3]{27x - 54} = 4$.
[Answer: 67]

Let $u = (x-2)^{1/3}$. Then the equation becomes $u^2 - 3u = 4$, or $u^2 - 3u - 4 = 0$. This can be solved to yield $u = 4$ and $u = -1$. Substituting back yields $(x-2)^{1/3} = 4$ and $(x-2)^{1/3} = -1$, which when solved individually yields $x = 66$ and $x = 1$, making the desired quantity $66 + 1 = 67$.

4-3 The function $f(x) = \sqrt{3x + k} - \sqrt{x^2 - 10x + 256}$ where k is a constant has a y-intercept of $(0,2)$, x-intercepts of $(a, 0)$ and $(b, 0)$, and a domain of $x \geq c$. Find the value of $|a| + |b| + |c|$.
[Answer: 129]

Setting $x = 0$ yields $f(0) = \sqrt{k} - \sqrt{256} = 2$, which means that $k = 18^2 = 324$. Setting $\sqrt{3x + 324} - \sqrt{x^2 - 10x + 256} = 0$, then bringing one of the radicals to the other side and squaring yields $3x + 324 = x^2 - 10x + 256$, or $x^2 - 13x - 68 = 0$, which can be solved by factoring to yield $x = 17$ and $x = -4$. Finally, note that $x^2 - 10x + 256 > 0$ for all x, so the domain will be determined by $3x + 324 \geq 0$, which yields $x \geq -108$. Therefore the desired quantity is $17 + 4 + 108 = 129$.

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<u>Round 5: Polynomials and Advanced Factoring</u>

5-1 A quadratic has the form $ax^2 - bx + b = 0$ where a and b are positive integers and that have no common factors greater than 1. If $x = 20$ is a solution of the quadratic, find $a + b$.
[Answer: 419]

Setting $x = 20$ yields $400a - 20b + b = 0$, or $400a = 19b$. In order for a and b to be coprime positive integers, it follows that $a = 19$ and $b = 400$, making the desired quantity $19 + 400 = 419$.

5-2 Let $f(x)$ be a polynomial function that contains the points $(1,2)$, $(2,4)$, $(3,8)$, and $(4,17)$. If $n > 2$ is the degree of $f(x)$ and is the smallest degree possible, find the value of $n + f(6)$.
[Answer: 65]

Since there are four points provided, the smallest degree possible that could fit them is $n = 3$. One way to solve this is to use differences since all the x-values produce an arithmetic sequence. The first differences in the y-values are $2, 4,$ and 9. The second differences are 2 and 5, and the third difference is 3. Assuming the third differences are constant yields additional second differences of 8 and 11 and additional first differences of 17 and 28. This means that $f(5) = 17 + 17 = 34$ and $f(6) = 34 + 28 = 62$, making the desired quantity $3 + 62 = 65$.

5-3 What is the sum of the squares of the three complex zeros of
$f(x) = x^3 - 19x^2 + 83x - 2025$?
[Answer: 195]

Let the three complex zeros be $a, b,$ and c. Note that $a + b + c = 19$, and $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc) = 361$. Finally, note that $ab + ac + bc = 83$, yielding $a^2 + b^2 + c^2 + 2(83) = 361$, making the desired quantity $361 - 2(83) = 195$.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
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Round 6: Counting and Probability

6-1 The local weather man loves probability. He says that the weather on each day this weekend will be either sunny or rainy. There is a $\frac{3}{5}$ probability that it will rain on Saturday. If it rains on Saturday there is a $\frac{2}{3}$ probability it will rain on Sunday as well. If it does not rain on Saturday, there is a $\frac{1}{4}$ probability it will rain on Sunday. The probability that it will be sunny both days is $\frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 13]

The probability that it will be sunny on Saturday is $\frac{2}{5}$, and if it is sunny on Saturday, the probability it will also be sunny on Sunday is $\frac{3}{4}$. Therefore the probability it will be sunny both days is $\frac{2}{5} * \frac{3}{4} = \frac{3}{10}$, making the desired quantity $3 + 10 = 13$.

6-2 Consider the weatherman's forecast from 6-1. If it is rainy on at least one day, the probability that it will be rainy on both days is $\frac{p}{q}$ where p and q are positive integers with no common factors greater than 1. Find $p + q$.
[Answer: 11]

This probability is found by computing $\frac{P(rainy\ both\ days)}{P(rainy\ on\ at\ least\ one\ day)}$, which is $\frac{\frac{3}{5}*\frac{2}{3}}{\frac{3}{5}+\frac{2}{5}*\frac{1}{4}} = \frac{4}{7}$, making the desired quantity $4 + 7 = 11$.

6-3 An AP Statistics class has 20 students, including k juniors. The students line up single file to collect their textbook. If the students line up in random order and the third student in line is a junior, the probability that the student is the first junior in line is P. Find the value of k that minimizes $\left|P - \frac{1}{2}\right|$.
[Answer: 6]

If there are k juniors, there are $20 - k$ non-juniors. Let this quantity equal n. We need the probability that the first two students in line are non-juniors given that the third student is a junior. This probability can be computed as $\frac{n}{19} * \frac{n-1}{18} = \frac{n(n-1)}{342}$. We therefore need a value of n such that $n(n-1)$ is as close as possible to $\frac{1}{2}(342) = 171$. Note that $(15)(14) = 210$,

$(14)(13) = 182$, and $13(12) = 156$. Therefore $n = 14$ produced the closest result, making the desired quantity $k = 20 - 14 = 6$.

FAIRFIELD COUNTY MATH LEAGUE 2024-2025
Match 6
Team Round

Please write your answers on the answer sheet provided.

1. A particular n-gon has 3 right angles and the rest of its angles measure $179°$. What is the value of n?
[Answer: 93]

Setting up $3(90) + (n-3)(179) = (n-2)(180)$ yields $270 + 179n - 537 = 180n - 360$, which yields $n = 93$.

2. If for all $x \neq 0$, $y \neq 0$, and $z \neq \frac{1}{2}$, $\frac{1}{x} = \frac{3}{2y} - \frac{5}{6-\frac{3}{z}}$, then $y = \frac{Axz-Bx}{Cxz+Dz-E}$, where $A, B, C, D,$ and E are positive integers that have no factors common to all of them greater than 1. Additionally, there is a positive integer $F > 2$ such that $z \neq \frac{1}{F}$ if $x = 3$. Find $A + B + C + D + E + F$.
[Answer: 62]

Setting $\frac{3}{2y} = \frac{1}{x} + \frac{5z}{6z-3} = \frac{5xz+6z-3}{6xz-3x}$ then yields $y = \frac{18xz-9x}{10xz+12z-6}$. Since $10xz + 12z - 6 \neq 0$, letting $x = 3$ yields $30z + 12z - 6 \neq 0$, or $z \neq \frac{1}{7}$. Therefore the desired quantity is $18 + 9 + 10 + 12 + 6 + 7 = 62$.

3. Consider a right square pyramid with a base with edge length of 12 and a volume of $288\sqrt{2}$ cubic units. Point A lies on the midpoint of one of the edges of the base, and point B lies on the face opposite of A and is equidistant from all three vertices of its face. Find $(AB)^2$.
[Answer: 108]

Using $V = \frac{1}{3}b^2h$ where b is the base edge length and h is the height, we can solve for h to find $h = 6\sqrt{2}$. We can see that the slant height of a triangular face will be $\sqrt{6^2 + (6\sqrt{2})^2} = 6\sqrt{3}$ units, and since the faces have bases of length 12 and heights of $6\sqrt{3}$, the faces are equilateral triangles. That means that point B, which would technically be the circumcenter of the triangle, lies $\frac{1}{3}$ up the slant height of the triangular faces. The right triangle formed by half of the base length and the pyramid height (with legs of 6 and $6\sqrt{2}$) is similar to the right triangle with B as one of its vertices corresponding to the vertex at the top of the pyramid (with legs of 2 and $2\sqrt{2}$). Therefore the distance AB is the length of the hypotenuse of a right triangle with legs of 10 and $2\sqrt{2}$, making the desired quantity $10^2 + (2\sqrt{2})^2 = 108$.

4. The equation $\left(\sqrt{x}-\frac{6}{\sqrt{x}}\right)\left(\sqrt{x}-\frac{15}{\sqrt{x}}\right)=\frac{7}{2}$ has positive solutions $x=a$ and $x=b$ where $a<b$. Find $10a+b$.
[Answer: 65]

Expanding the product on the left side yields $x-21+\frac{90}{x}=\frac{7}{2}$, or $2x^2-49x+180=0$. This is factorable into $(x-20)(2x-9)=0$, making the solutions $x=\frac{9}{2}$ and $x=20$. Therefore the desired quantity is $10\left(\frac{9}{2}\right)+20=45+20=65$.

5. How many of zeros of the polynomial $f(x)=x^{2025}+2024x^{2024}-2025$ will be complex non-real?
[Answer: 2022]

Using Descartes' Law of Signs, there must be 1 positive real zero, and inspection shows that $x=1$ is a zero of $f(x)$. Likewise there must be 2 or 0 negative real zeros. Writing $f(x)=x^{2024}(x+2024)-2025$ shows that there must be a sign change in the interval $-2024<x<-2023$ as well as $-2<x<-1$, revealing that there are 2 negative real zeros. Therefore the number of complex non-real zeros must be $2025-1-2=2022$.

6. A box contains 30 fair coins of identical size and weight. Ten coins have heads on both sides, ten coins have tails on both sides, and ten coins have heads on one side and tails on the other. One coin is picked at random from the box and flipped twice. If both flips come up heads, the probability the coin has heads on both sides is k%. Find the value of k.
[Answer: 80]

This probability is found by $\frac{P(heads\ on\ both\ sides\ and\ heads\ twice)}{P(heads\ twice)}$. Since the probability that a coin chosen at random from the 30 coins has heads on both sides is $\frac{1}{3}$, the numerator will be $\frac{1}{3}(1)(1)=\frac{1}{3}$, and the denominator will consist of $P(heads\ twice\ and\ heads\ on\ both\ sides)+P(heads\ twice\ and\ heads\ on\ one\ side)$, which is $\frac{1}{3}(1)(1)+\frac{1}{3}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\frac{1}{3}+\frac{1}{12}=\frac{5}{12}$. Therefore the desired probability is $\frac{\frac{1}{3}}{\frac{5}{12}}=\frac{4}{5}$, or 80%.