FAIRFIELD COUNTY MATH LEAGUE 2024-2025
Match 5
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Fractions and Exponents

1-1 Mr. Finebar has $n < 200$ peanuts. He hands one half of his peanuts to Mr. Krunchle, one third of his peanuts to Ms. Crysp, and one eighth of his peanuts to Ms. Gheefinger. If Mr. Finebar has an odd whole number of peanuts left over, find the largest possible value of n.
[Answer: 168]

The fraction of peanuts given is $\frac{1}{2}+\frac{1}{3}+\frac{1}{8}=\frac{23}{24}$, leaving $\frac{1}{24}$ of the peanuts remaining, so $n = 24k$. If $k = 7$, $n = 168$, but if $k = 9$, $n = 216$, which is too large, making the desired value 168.

1-2 There are two single digit numbers, a and b where $a > b$, that are the most probable units digits of p^q where p and q are positive integers less than or equal to 100. Find $10a + b$.
[Answer: 61]

All units digits of successive positive integer powers can all be described in cycles of 4; for numbers ending in 1: 1, 1, 1, 1. 2: 2, 4, 8, 6. With 10 possible units digits and 4 (not-necessarily distinct) units digits coming from integer powers each, there are 40 total possibilities. 6 and 1 appear the most often with 8 total occurrences, making the desired quantity 61.

1-3 How many zeros are found in the whole number portion of the quotient $\frac{10^{2025}+10^{24}}{10^{20}-1}$?
[Answer: 1904]

Computing the quotient using long division begins to produce a quotient starting with $10^{2005} + 10^{1985} + 10^{1965} + \cdots$, which will continue until the terms $10^5 + 10^4$, at which point all of the terms will consist of 10 raised to a negative power, the first of which being 10^{-15}. Since the leading term is 10^{2005}, there are 2006 total digits in the whole number portion of the quotient. There are 101 total digits of 1 from 10^{2005} down to 10^5, and then one last digit of 1 for 10^4. This makes the total number of zeros $2006 - 102 = 1904$.

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Round 2: Rational Expressions and Equations

2-1 The expression $20 + \frac{1}{24+\frac{1}{25+x}}$ is equivalent to $\frac{ax+b}{cx+d}$ for positive integers $a, b, c,$ and d, which as a set have no common factors greater than 1. What is the value of a?
[Answer: 481]

The expression can be simplified as $20 + \frac{1}{\frac{24x+600}{x+25}} = 20 + \frac{x+25}{24x+600}$, which means that when the entire expression is put over the denominator, the coefficient of x in the numerator becomes $20 * 24 + 1 = 481$.

2-2 Find the smallest positive integer n such that $\sum_{k=2}^{n} \frac{2}{(2k-1)(2k-3)} > \frac{99}{100}$.
[Answer: 51]

Note that $\frac{2}{(2k-1)(2k-3)} = \frac{2k-1-(2k-3)}{(2k-1)(2k-3)} = \frac{1}{2k-3} - \frac{1}{2k-1}$. This makes $\sum_{k=2}^{n} \frac{2}{(2k-1)(2k-3)} = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2n-3} + \frac{1}{2n-1}$, which simplifies to $1 - \frac{1}{2n-1}$. Therefore we can set up $1 - \frac{1}{2n-1} > \frac{99}{100}$, so $\frac{1}{2n-1} < \frac{1}{100}$, making $2n - 1 > 100$. Thus $n > 50.5$, making the desired quantity 51.

2-3 The equation $x + \frac{6}{x-4} = \frac{x+2}{x-4}$ has extraneous solution $x = a$ and non-extraneous solution $x = b$. The equation $x^2 + \frac{px^2-37x+q}{x^2-16x+r} = 2x$, where p, q, and r are positive integers, has extraneous solution $x = b$ and non-extraneous solution $x = a$. Find $p + q + r$.
[Answer: 52]

Note that $x = 4$ must be the extraneous solution in the first equation. The non-extraneous solution is found by solving $x^2 - 4x + 6 = x + 2$, or $x^2 - 5x + 4 = 0$, making the non-extraneous solution $x = 1$. The means $r = 15$ so that $x = 1$ makes the expression in the denominator in the second equation equal to zero. This also means the value $x = 1$ would solve $x^2(x-1) + \frac{px^2-37x+q}{x-15} = 2x(x-1)$, so letting $x = 1$ in this equation yields $\frac{p-37+q}{-14} = 0$. Therefore $p + q = 37$, making the desired quantity $37 + 15 = 52$.

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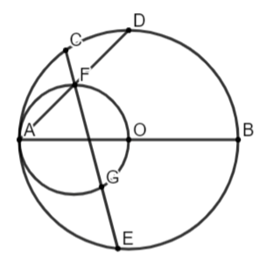
Round 3: Circles

3-1 Consider a circle with center Q and points U, A, and D on the circumference of the circle such that $QUAD$ is a convex quadrilateral. If $m\angle QUD = 20°$, find $m\angle UAD$ in degrees.
[Answer: 110]

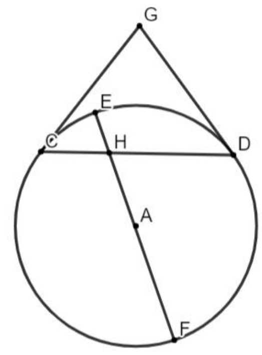
Since $\overline{QU}$ and $\overline{QD}$ are radii, triangle QUD is isosceles. This means $m\angle QDU = 20°$ and therefore $m\angle DQU = 140°$. This means $m\,\widehat{DAU} = 140°$, making the measure of the remaining arc $220°$, and therefore $m\angle DAU = 110°$.

3-2 See the diagram. Consider circle O with diameter $\overline{AB}$. Chords $\overline{CE}$ and $\overline{AD}$ meet at point F, and a circle with diameter $\overline{AO}$ intersects $\overline{CE}$ at F and G. If the length of $\widehat{AE}$ is 40π and the length of $\widehat{CD}$ is 12π, then the length of $\widehat{AG}$ is $n\pi$. Find the value of n.
[Answer: 26]



If circle O has radius r, and $m\angle AFE = \frac{m\widehat{AE}+m\widehat{CD}}{2}$, then $r * m\angle AFE = \frac{r*m\widehat{AE}+r*m\widehat{CD}}{2} = \frac{40\pi+12\pi}{2} = 26\pi$. This makes the desired arclength $\frac{r}{2} * (2m\angle AFG) = r * m\angle AFE = 26\pi$, making the desired value 26.

3-3 See the diagram. The circle has center A and radius r, tangents $\overline{CG}$ and $\overline{GD}$, and diameter $\overline{EF}$ which intersects chord $\overline{CD}$ at point H. If $CG = r^2 + 2025r$, $GD = 2025r + 21$, $AH = 3$, and CH and HD are integers, find the sum of all possible values of CD.
[Answer: 15]



Since the tangent segments are congruent, we know $r^2 + 2025r = 2025r + 21$, so $r^2 = 21$. Since $(HF)(EH) = (CH)(HD)$, we have $(r+3)(r-3) = (CH)(HD)$. This means $r^2 - 9 = (CH)(HD)$. Therefore $(CH)(HD) = 21 - 9 = 12$. Thus possible values of CD are sums of pairs of integers that multiply to 12: $4 + 3$, $6 + 2$, and $1 + 12$. However, since $4 < \sqrt{21} < 5$, $8 < 2\sqrt{21} < 10$, and therefore CD cannot equal 13. This makes the desired quantity $7 + 8 = 15$.

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Round 4: Quadratic Equations & Complex Numbers

4-1 If $z = 4 - 5i$ is a solution to $az^2 + bz + c = 0$, where $a, b,$ and c are integers with no common factors as a set greater than 1 and $a > 0$, find $|a| + |b| + |c|$.
[Answer: 50]

The other solution would have to be $4 + 5i$, since the coefficients are integers. This means the sum of the solutions $4 - 5i + (4 + 5i) = 8$ and the product of the zeros is $(4 - 5i)(4 + 5i) = 16 + 25 = 41$. This means the quadratic would be $z^2 - 8z + 41 = 0$, making the desired quantity $1 + 8 + 41 = 50$.

4-2 If $k = a + bi$, where a and b are positive integers, is a complex constant with the same modulus as the solutions to $z^2 + 12z + 2025 = 0$, find $a + b$.
[Answer: 63]

Note that the product of the zeros, which would have to be of the form $p \pm qi$, would be $p^2 + q^2$, which form the equation is 2025. This means the modulus of one of the zeros is 45. Therefore we require a Pythagorean triple of integers that has 45 has the largest number. Since $45 = 9(5)$, it follows that one Pythagorean triple would have to be $(9(3), 9(4), 9(5))$ or $(27{,}36{,}45)$, which is the only Pythagorean triple with 45 as the largest number. This makes the desired quantity $27 + 36 = 63$.

4-3 A quadratic function $f(z) = az^2 + bz + c$ where a, b, and c are complex constants, has the properties that $f(0) = f(3 + i) = 5 - 2i$ and $f(3 - i) = 7 - 16i$. Find $|a|^2$.
[Answer: 5]

We know $c = 5 - 2i$ since it is $f(0)$. Since $f(3 + i)$ has the same value, we can rewrite $f(z)$ as $f(z) = az\big(z - (3 + i)\big) + 5 - 2i$. Using the next piece of information gives $f(3 - i) = a(3 - i)\big(3 - i - (3 + i)\big) + 5 - 2i = a(3 - i)(-2i) + 5 - 2i = 7 - 16i$, which means $a(-6i - 2) = 2 - 14i$. Since $|2 - 14i|^2 = 4 + 196 = 200$ and $|(a)(-2 - 6i)|^2 = |a|^2(40)$, it follows that $|a|^2 = 5$.

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Match 5
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Round 5: Trigonometric Equations

5-1 If $\sin(x) = 9\cos(x)$, find $\sec^2(x)$.
[Answer: 82]

Since $\frac{\sin(x)}{\cos(x)} = \tan(x) = 9$, it follows that $\sec^2(x) = \tan^2(x) + 1 = 81 + 1 = 82$.

5-2 Find the sum of all possible values of $\tan(x)$ such that $4\sin^2(x) + 5 = 2025\sin(2x)$.
[Answer: 450]

Using double angle formulas, we get $4\sin^2(x) + 5 = 4050\sin(x)\cos(x)$. Dividing every term by $\cos^2(x)$ yields $4\tan^2(x) + 5\sec^2(x) = 5050\tan(x)$. Using $\sec^2(x) = 1 + \tan^2(x)$ and rearranging, we get $9\tan^2(x) - 5050\tan(x) + 5 = 0$, or $\tan^2(x) - 450\tan(x) + \frac{5}{9} = 0$. As this equation is quadratic in $\tan(x)$, the sum of all possible values of $\tan(x)$ is 450.

5-3 If A is an angle in quadrant one such that $\sin\left(A + \frac{\pi}{6}\right) = \frac{\sqrt{6}}{4}$, then $\tan(A) = \frac{\sqrt{a} - b\sqrt{c}}{d}$, where a, b, and c are positive integers and a and c have no perfect square factors greater than 1. Find $a + b + c + d$.
[Answer: 23]

Expanding the left side using identities produces $\sin(A)\cos\left(\frac{\pi}{6}\right) + \cos(A)\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{6}}{4}$. This means $\frac{\sqrt{3}}{2}\sin(A) + \frac{1}{2}\cos(A) = \frac{\sqrt{6}}{4}$. Multiplying by $2\sec(A)$ yields $\sqrt{3}\tan(A) + 1 = \frac{\sqrt{6}}{2}\sec(A)$, and squaring both sides makes $3\tan^2(A) + 2\sqrt{3}\tan(A) + 1 = \frac{3}{2}\sec^2(A)$. Using $\sec^2(x) = 1 + \tan^2(x)$ and rearranging, we get $\frac{3}{2}\tan^2(x) + 2\sqrt{3}\tan(A) - \frac{1}{2} = 0$. Using the quadratic formula produces $\tan(A) = \frac{-2\sqrt{3} \pm \sqrt{15}}{3}$, but for $\tan(A)$ to be positive, we have $\tan(A) = \frac{\sqrt{15} - 2\sqrt{3}}{3}$, making the desired quantity $15 + 2 + 3 + 3 = 23$.

FAIRFIELD COUNTY MATH LEAGUE 2024-2025
Match 5
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Please write your answers on the answer sheet provided.

Round 6: Sequences & Series

6-1 A sequence is defined as: $a_1 = 11$, $a_2 = 14$, and for all $n > 2$, $a_n = a_{n-1} - a_{n-2}$. What is the sum of the first 2025 terms of the sequence?
[Answer: 28]

The first few terms of the sequence will be $11,14,3,-11,-14,-3,11,14,\ldots$ and so this will create a pattern that cycles every six terms that will add to zero. Note that 2025 divided by 6 produces 337 with a remainder of 3, making the last three terms $11,14,3$. Since all of the prior terms added together produce zero, the sum will be $11 + 14 + 3 = 28$.

6-2 The sum of the first two terms of an infinite geometric series is 30 and the sum of the entire series is 54. Find the sum of all possible values of the first term of the series.
[Answer: 108]

Since $a + ar = a(1 + r) = 30$ and $\frac{a}{1-r} = 54$, we have $\frac{30}{(1+r)(1-r)} = 54$, so $1 - r^2 = \frac{30}{54} = \frac{5}{9}$, so $r^2 = \frac{4}{9}$, making $r = \pm\frac{2}{3}$. If $r = \frac{2}{3}$, then $a = 30\left(\frac{3}{5}\right) = 18$, and if $r = -\frac{2}{3}$, $a = 30(3) = 90$, making the desired quantity $18 + 90 = 108$.

6-3 The sum of the first six terms of a decreasing arithmetic series of integers is 9. If the first term is less than 100, find the largest possible integer value of the third term in the series.
[Answer: 21]

Let a be the first term in the series and let d be the positive common difference between successive terms. The sum of the first six terms will be $a + (a - d) + (a - 2d) + (a - 3d) + (a - 4d) + (a - 5d) = 6a - 15d$. Therefore $6a - 15d = 9$, and so $2a - 5d = 3$. Note that $(4,1)$ is a solution (a, d) to the equation. The largest value of a less than 100 that is a part of a solution is $4 + 19(5) = 99$, which means the corresponding value of d will be $1 + 19(2) = 39$. The third term is therefore $99 - 2(39) = 99 - 78 = 21$.

FAIRFIELD COUNTY MATH LEAGUE 2024-2025
Match 5
Team Round

Please write your answers on the answer sheet provided.

T-1 For each rational number N there is a sequence of n strictly increasing positive integers $x_1, x_2, \dots, x_n$ such that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = N$ and each integer x_a is the smallest possible integer such that $\frac{1}{x_1} + \cdots + \frac{1}{x_a} \leq N$. If $N = \frac{2024}{2025}$, find x_5.
[Answer: 16700]

Note that $N = 1 - \frac{1}{2025}$. To start, $x_1 = 2$ and $x_2 = 3$. Since $\frac{1}{2} + \frac{1}{3} = 1 - \frac{1}{6}$, $x_3 = 7$, bringing the sum to $1 - \left(\frac{1}{6} - \frac{1}{7}\right) = 1 - \frac{1}{42}$. Thus $x_4 = 43$, bringing the sum to $1 - \left(\frac{1}{42} - \frac{1}{43}\right) = 1 - \frac{1}{1806}$. x_5 cannot be 1807 since the resulting sum will be greater than N. To find the value of x_5, compute $\frac{1}{1806} - \frac{1}{2025} = \frac{219}{(1806)(2025)} = \frac{73}{1219050}$. Since we can use division to show that $\frac{1}{16700} < \frac{73}{1219050} < \frac{1}{16699}$, it follows that $x_5 = 16700$.

T-2 What is the sum of all single digit positive integers k such that the remainder when $x^{2025} + k$ is divided by $x - k$ has a units digit of 8?
[Answer: 13]

Since the remainder will be $k^{2025} + k$, and since 2025 is one more than a multiple of 4 and all units digits of natural number powers of natural numbers can be thought of through cycles of 1, 2, or 4, the units digit of the remainder will be the units digit of the result of $k + k$. This will have a units digit of 8 when $k = 4$ and $k = 9$, making the desired quantity $4 + 9 = 13$.

T-3 Consider a circle with diameter $\overline{AB}$, passing through center O, with chord $\overline{CD}$ intersecting $\overline{AB}$ at point E such that $CE: EO: ED = 1: 2: 3$. If the area of the circle is $n\pi$ where n is a positive integer less than 1000, find the largest possible integer value of CE.
[Answer: 11]

Let $AO = r$ and $CE = x$, which makes $EO = 2x$ and $ED = 3x$ (and consequently $CD = 4x$). Since E is the intersection of chords $\overline{AB}$ and $\overline{CD}$, it follows that $(AE)(EB) = (CE)(ED)$, or
$(r + 2x)(r - 2x) = x(3x)$, or $r^2 - 4x^2 = 3x^2$. Consequently $r^2 = 7x^2 = n$, and the largest integer value of x such that $7x^2 < 1000$ is 11.

T-4 The complex number $(1 + i)^{2025} = k + ki$, where k is a positive integer. If $\log(2) \approx .301$, how many digits are in k?
[Answer: 305]

Note that $|1 + i| = \sqrt{2}$ and $|k + ki| = k\sqrt{2}$. Since $|(1 + i)^{2025}| = \sqrt{2}^{2025} = k\sqrt{2}$, it follows that $k = \sqrt{2}^{2024} = 2^{1012}$. Since the digits of a whole number n is the ceiling of $\log(n)$, we have $\log(k) = \log(2^{1012}) = 1012\log(2) \approx 1012(.301) \approx 304.6$, it follows that the number of digits in k is 305.

T-5 For how many positive integers N is there an angle x such that $\sec(2x) = N$ and $.36 < \sin^2(x) < .49$?
[Answer: 46]

We have $\cos(2x) = \frac{1}{N}$, so $1 - 2\sin^2(x) = \frac{1}{N}$, and so $\sin^2(x) = \frac{1}{2}\left(1 - \frac{1}{N}\right) = \frac{N-1}{2N}$. Note that when $N = 50$ this quantity is $\frac{49}{100}$, so N will be less than 50. For $\frac{N-1}{2N} > .36$, we have $.28N > 1$, so $N \geq 4$. Therefore
$4 \leq N \leq 49$, making 46 possible values of N.

T-6 An arithmetic sequence $a_1, a_2, \ldots$ contains the terms $a_n = n$ and $a_{n+1} = -1$. For how many values of n, $1 \leq n \leq 2025$, is a_1 a multiple of 5?
[Answer: 405]

Note that if d is the common difference between the terms of the sequence, then $a_n = a_1 + (n-1)d = n$, and $a_{n+1} = n + d = -1$, we have $d = -1 - n$. This means $n = a_1 + (n-1)(-1-n)$, so $n = a_1 - n^2 + 1$, or $a_1 = n^2 + n - 1$. This means $a_1 = n(n+1) - 1$. Since there is no positive integer n such that $n(n+1)$ has a units digit of 1 as the product must be even, we need numbers such that $n(n+1)$ ends with a units digit of 6. This will occur when n has a units digit of 2 or 7. This means there are two such values of n for every 10 numbers, making the total number of possible values $\frac{2020}{10}(2) + 1$ (with the last value being $n = 2022$) for a total of 405 possible values.