**FAIRFIELD COUNTY MATH LEAGUE 2024–2025**
**Match 4**
Individual Section

**Please write your answers on the answer sheet provided.**

Round 1: Basic Statistics

1-1 A set of five positive integers has a median of 8, a unique mode of 2, and a range of 13. If the arithmetic mean of the numbers is an integer, what is the sum of the numbers?

1-2 The Mean Absolute Deviation (MAD) of a set of numbers is the arithmetic mean of the absolute difference of each number and the arithmetic mean of the set. A set of four integers $a, b, c,$ and $d$ has an arithmetic mean $\bar{x}$. If $a < b < \bar{x} < c < d$, $\bar{x} = 26$ and the MAD of the set is 15, find the largest possible value of $d$.

1-3 Set A consists of 50 positive 1-digit, 2-digit, and 3-digit integers. The elements of set B are formed by placing the digit "1" in front of each element of set A. The arithmetic mean of the elements of set B is 409.6 greater than the arithmetic mean of the elements of set A. Find the largest possible number of total digits of all the elements of set A.

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Round 2: Quadratic Equations

2-1 The quadratic function $f(x) = x^2 - (3k + 1)x + 64$ has only one distinct positive real zero. Find $f(k)$.

2-2 The quadratic equation $4x^2 - 12x + 7 = 0$ has zeros $x = m$ and $x = n$. The quadratic equation $ax^2 + bx + c = 0$, where $a, b,$ and $c$ are relatively prime integers and $a > 0$, has zeros of $x = 2m + 1$ and $x = 2n + 1$. Find $|a| + |b| + |c|$.

2-3 There are two positive values of $p$ such that the equations $y = 3x - 2$ and $x = py^2 + 4py + 4$ share only one solution $(x, y)$. The larger of the two values of $p$ is $\frac{a+\sqrt{b}}{c}$, where $a, b,$ and $c$ are positive integers and $b$ has no perfect square factors greater than 1. Find $a + b + c$.

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Round 3: Similarity

3-1 Ivana Nicegarten is filling her decorative pool, which is shaped like a cone with the vertex pointed downward into the ground. Water is pouring into the pool at a constant rate. After 12 minutes, the water in the pool has depth $d$. How many total minutes will it take for the depth to be $2d$?

3-2 Consider trapezoid $GEOM$, with $\overline{GE}||\overline{OM}$, angles $G$ and $M$ are right angles, and $m\angle GME = m\angle EOM$. If $GM = 12$ and $GE = 16$, find the perimeter of the trapezoid.

3-3 Consider parallel lines $l_1$ and $l_2$. One transversal intersects $l_1$ at $A$ and $l_2$ at $B$. A second transversal intersects $l_1$ at $C$ and $l_2$ at $D$, and the two transversals intersect at point $E$ which is between points $C$ and $D$. If $AC = 12$, $CE = 8$, $DE = 6$, and the total area of triangles $ACE$ and $DBE$ is $T$ square units, then the distance between lines $l_1$ and $l_2$ is $\frac{a}{b}T$ where $a$ and $b$ are positive integers with no common factors greater than 1. Find $a + b$.

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Round 4: Variation

4-1 If $y$ varies directly as $x$ and $x = 40$ when $y = 50$, find the value of $x$ when $y = 95$.

4-2 A relationship where $z$ varies directly as the 1.5 power of $x$ and inversely as the square of $y$ contains the ordered triple $(p, q, r)$. Increasing $p$ by $300\%$ and decreasing $q$ by $33\frac{1}{3}\%$ produces the new $z$-value $s$. Find $\frac{s}{r}$.

4-3 If $y$ varies inversely as the $n$th power of $x$ for some positive number $n$, and $y = 32$ when $x = 1$ and $y = 1$ when $x = 4$, then $y = k$ when $x = 36$. Find $\frac{1}{k}$.

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**Match 4**
Individual Section

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Round 5: Trig Expressions & DeMoivre's Theorem

5-1 A complex number $z_1$ has an argument of $342°$ and is one of the complex $n$-th roots of the complex non-real number $z_2$. If $z_2$ has no complex $n$-th roots with a negative imaginary component and a real component greater than the real component of $z_1$, find the largest possible value of $n$.

5-2 If $z = \left(\frac{1+7i}{2+bi}\right)^4$ where $b$ is a real number and $z$ has the same modulus as $3.2 + 2.4i$, find $b^2$.

5-3 If $A$ is an angle such that $\sin\left(A + \frac{3\pi}{4}\right) = \frac{\sqrt{5}}{8}$, then $\sin(2A) = \frac{a}{b}$ where $a$ and $b$ are positive integers with no common factors greater than 1. Find $a + b$.

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**Match 4**
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Round 6: Conic Sections

6-1 One of the foci of the ellipse $\frac{(x-3)^2}{50}+\frac{(y+7)^2}{14}=1$ has a positive $x$-coordinate $a$. What is the value of $a$?

6-2 A hyperbola has an asymptote with an equation of $y=\frac{1}{2}x+\frac{7}{2}$ and a range of $(-\infty,2]\cup[6,\infty)$. The largest $y$-value of the hyperbola where $x=4$ is $\frac{a}{b}$ where $a$ and $b$ are positive integers with no common factors greater than 1. Find $a+b$.

6-3 An ellipse is centered at origin and has a focus at $\left(0,\sqrt{2}\right)$ and an area of $4\sqrt{3}\pi$. A circle is also centered at the origin and intersects the ellipse at points that lie on $y=x$ and $y=-x$. The square of the radius of the circle is $\frac{m}{n}$ where $m$ and $n$ are positive integers with no common factors greater than 1. Find $m+n$. (Note: the area of an ellipse is $\pi ab$ where $a$ and $b$ are semi-major and semi-minor axis lengths.)

**FAIRFIELD COUNTY MATH LEAGUE 2024-2025**
**Match 4**
Team Round

**Please write your answers on the answer sheet provided.**

1. The geometric mean of a set of $n$ numbers is the $n$th root of the product of the numbers. A set of three distinct positive integers greater than 1 has the property that its arithmetic and geometric means are both integers. What is the smallest possible value of the arithmetic mean of the set?

2. The quadratic equations $x^2 - 8x + p = 0$ and $x^2 - 2x + q = 0$, where $p$ and $q$ are real constants, each have two positive solutions for $x$. They share one solution and the other solutions are reciprocals of each other. If $p = a + b\sqrt{c}$, where $a, b,$ and $c$ are positive integers and $c$ has no perfect square factors greater than 1, find $a + b + c$.

3. Consider rectangle $FCML$, with $FC = 8$ and $CM = 10$. Point $N$ is draw on diagonal $\overline{FM}$ such that the distance from $N$ to $\overline{ML}$ is 7. Point $T$ is drawn on $\overline{FL}$ such that $\overline{CT}$ contains point $N$. $LT = \frac{a}{b}$, where $a$ and $b$ are positive integers with no common factors greater than 1. Find $a + b$.

4. If $y$ varies directly as the second power of $x$ and the ordered triple $(a, b, c)$ has the properties that $a, b,$ and $c$ are all different positive integers and both $(a, b)$ and $(b, c)$ fit this particular variation relationship, find the smallest possible value of $a + b + c$ when $b = 30$.

5. If $k$ is a positive number such that $\arctan\left(\frac{1}{3}\right) + \arctan(k) = \arctan\left(\frac{2}{3}\right)$, then $k = \frac{a}{b}$ where $a$ and $b$ are positive integers with no common factors greater than 1. Find $a + b$.

6. The circle $x^2 + y^2 - 16x - 6y + 48 = 0$ and the line $x = k$, which lies to the left of the center of the circle, intersect at points $A$ and $B$ such that $AB = 8$. A particular conic section represents the set of all points equidistant from the center of the circle and the line. Find the $x$-coordinate of the two intersection points between the conic section and the circle.