FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 4
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Basic Statistics

1-1 A set of five positive integers has a median of 8, a unique mode of 2, and a range of 13. If the arithmetic mean of the numbers is an integer, what is the sum of the numbers?
[Answer: 40]

From the given information, the five numbers must be $\{2,2,8,x,15\}$, where x is unknown. Since $8 < x < 15$ (as there is only one mode) and $27 + x$ must be a multiple of 5, x must have a value of 13, making the sum of the numbers 40.

1-2 The Mean Absolute Deviation (MAD) of a set of numbers is the arithmetic mean of the absolute difference of each number and the arithmetic mean of the set. A set of four integers $a, b, c,$ and d has an arithmetic mean $\bar{x}$. If $a < b < \bar{x} < c < d$, $\bar{x} = 26$ and the MAD of the set is 15, find the largest possible value of d.
[Answer: 55]

One way to solve this is to set up $a + b + c + d = 26(4) = 104$, and $\frac{26-a+26-b+c-26+d-26}{4} = \frac{c+d-a-b}{4} = 15$, so $c + d - a - b = 60$. Combining the equations yields $2c + 2d + 164$, or $c + d = 82$. Since the smallest possible value of c is 27, the largest possible value of d is 55.

Set A consists of 50 positive 1-digit, 2-digit, and 3-digit integers. The elements of set B are formed by placing the digit "1" in front of each element of set A. The arithmetic mean of the elements of set B is 409.6 greater than the arithmetic mean of the elements of set A. Find the largest possible number of total digits of all the elements of set A.
[Answer: 110]

Let x be the number of 3-digit integers, y represent the number of 2-digit integers, and z represent the number of 1-digit integers. From the problem we have $1000x + 100y + 10z = 50(409.6) = 20480$, so $100x + 10y + z = 2048$. We also know $x + y + z = 50$. Subtracting the equations yields $99x + 9y = 1998$, or $11x + y = 222$, so $y = 222 - 11x$. This means the largest possible value of x is 20. Therefore possible values of (x, y, z) are $(20, 2, 28)$, $(19, 13, 18)$, and $(18, 24, 8)$. The total number of digits must be $3x + 2y + z$, or $3x + 2(222 - 11x) + (50 - x - (222 - 11x))$, which simplifies to $272 - 9x$, so the greatest number occurs with the smallest value of x, which is 18. Therefore the desired answer is $272 - 9(18) = 110$.

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Round 2: Quadratic Equations

2-1 The quadratic function $f(x) = x^2 - (3k+1)x + 64$ has only one distinct positive real zero. Find $f(k)$.
[Answer: 9]

To have one distinct positive real zero, $f(x) = (x-h)^2 = x^2 - 2hx + h^2$, and since $h^2 = 64$, we know $h = 8$ and therefore $3k + 1 = 16$, so $k = 5$ and therefore $f(5) = 5^2 - 16(5) + 64 = 9$.

2-2 The quadratic equation $4x^2 - 12x + 7 = 0$ has zeros $x = m$ and $x = n$. The quadratic equation $ax^2 + bx + c = 0$, where $a, b,$ and c are relatively prime integers and $a > 0$, has zeros of $x = 2m + 1$ and $x = 2n + 1$. Find $|a| + |b| + |c|$.
[Answer: 23]

If $4m^2 - 12m + 7 = 0$ and $x = 2m + 1$, then $m = \left(\frac{x-1}{2}\right)^2$. This makes the quadratic $4\left(\frac{x-1}{2}\right)^2 - 12\left(\frac{x-1}{2}\right) + 7 = 0$, which simplifies to $(x-1)^2 - 6(x-1) + 7 = 0$, which when expanded and simplified becomes $x^2 - 8x + 14 = 0$. This makes the desired quantity $1 + 8 + 14 = 23$.

2-3 There are two positive values of p such that the equations $y = 3x - 2$ and $x = py^2 + 4py + 4$ share only one solution (x, y). The larger of the two values of p is $\frac{a+\sqrt{b}}{c}$, where $a, b,$ and c are positive integers and b has no perfect square factors greater than 1. Find $a + b + c$.
[Answer: 53]

Substitution produces $x = p(3x-2)^2 + 4p(3x-2) + 4$, which when expanded and combined yields $9px^2 - x + 4 - 4p = 0$. To have only one real solution, the discriminant must equal zero, so $(-1)^2 - 4(4-4p)(9p) = 0$. Expanded and simplified, this becomes $144p^2 - 144p + 1 = 0$. This makes $p = \frac{144\pm\sqrt{144^2-4(144)}}{2*144} = \frac{144\pm12\sqrt{144-4}}{2*144} = \frac{144\pm24\sqrt{35}}{288} = \frac{6\pm\sqrt{35}}{12}$, making the desired quantity $6 + 35 + 12 = 53$.

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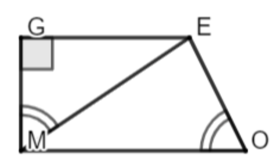
Round 3: Similarity

3-1 Ivana Nicegarten is filling her decorative pool, which is shaped like a cone with the vertex pointed downward into the ground. Water is pouring into the pool at a constant rate. After 12 minutes, the water in the pool has depth d. How many total minutes will it take for the depth to be $2d$?
[Answer: 96]

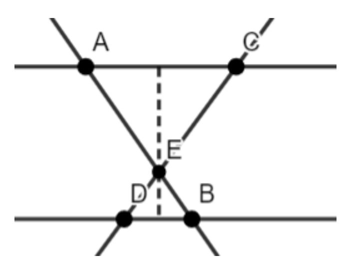
The cones with depth (or height) d and $2d$ are similar. Since the cone with depth $2d$ has twice the height, it will have $2^3 = 8$ times the volume. Therefore it will take 8 times longer to fill at a constant rate, making the desired quantity $12 * 8 = 96$.

3-2 Consider trapezoid $GEOM$, with $\overline{GE} || \overline{OM}$, angles G and M are right angles, and $m\angle GME = m\angle EOM$. If $GM = 12$ and $GE = 16$, find the perimeter of the trapezoid.
[Answer: 68]



See the diagram (not drawn to scale). Since both $\angle GEM$ and $\angle EMO$ are complimentary to angle $\angle GME$, they are congruent, and therefore triangle GME is similar to triangle EOM. Since $ME^2 = 12^2 + 16^2$, we have $ME = 20$. Also, $\frac{GE}{GM} = \frac{ME}{OE}$, so $\frac{16}{12} = \frac{20}{OE}$, and thus $OE = 15$, and using either similarity or pythagorean theorem, $OM = 25$. Therfore the desired quantity is $12 + 16 + 15 + 25 = 68$.

3-3 Consider parallel lines l_1 and l_2. One transversal intersects l_1 at A and l_2 at B. A second transversal intersects l_1 at C and l_2 at D, and the two transversals intersect at point E which is between points C and D. If $AC = 12$, $CE = 8$, $DE = 6$, and the total area of triangles ACE and DBE is T square units, then the distance between lines l_1 and l_2 is $\frac{a}{b}T$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 89]



See the diagram (not drawn to scale). Note that triangle EAC is similar to triangle EBD. Since $\frac{CE}{DE} = \frac{8}{6} = \frac{AC}{BD}$, it follows that $BD = 9$. Also the altitude of EBD from point E (h_1) is $\frac{3}{4}$ the length of the altitude of EAC from point E (h_2). Therefore $T = \frac{1}{2}(12)(h_1) + \frac{1}{2}(9)(h_2) = \frac{1}{2}(12)(h_1) + \frac{1}{2}(9)\left(\frac{3}{4}h_1\right) = \frac{75}{8}h_1$, so

$h_1 = \frac{8}{75}T$. That means $h_1 + h_2 = \frac{8}{75}T + \frac{3}{4}\left(\frac{8}{75}\right)T = \frac{14}{75}T$, making the desired quantity $14 + 75 = 89$.

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Round 4: Variation

4-1 If y varies directly as x and $x = 40$ when $y = 50$, find the value of x when $y = 95$.
[Answer: 76]

Setting up $\frac{50}{40} = \frac{95}{x}$, we have $x = \frac{4}{5}(95) = 4(19) = 76$.

4-2 A relationship where z varies directly as the 1.5 power of x and inversely as the square of y contains the ordered triple (p, q, r). Increasing p by 300% and decreasing q by $33\frac{1}{3}\%$ produces the new z-value s. Find $\frac{s}{r}$.
[Answer: 18]

The new ordered pair will be $\left(4p, \frac{2}{3}q, s\right)$. Setting $\frac{rq^2}{p^{1.5}} = \frac{s\left(\frac{2}{3}q\right)^2}{(4p)^{1.5}} = \frac{4sq^2}{9*8p^{1.5}}$, so $\frac{s}{r} = \frac{72}{4} = 18$.

4-3 If y varies inversely as the nth power of x for some positive number n, and $y = 32$ when $x = 1$ and $y = 1$ when $x = 4$, then $y = k$ when $x = 36$. Find $\frac{1}{k}$.
[Answer: 243]

Setting $32(1)^n = (1)4^n$ yields $n = \frac{5}{2}$. Therefore $32 = k(36)^{\frac{5}{2}} = 6^5$, and $k = \frac{2^5}{6^5} = \frac{1}{3^5}$, making the desired quantity 3^5 or 243.

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Round 5: Trig Expressions & DeMoivre's Theorem

5-1 A complex number z_1 has an argument of $342°$ and is one of the complex n-th roots of the complex non-real number z_2. If z_2 has no complex n-th roots with a negative imaginary component and a real component greater than the real component of z_1, find the largest possible value of n.
[Answer: 19]

Since all the n-th roots of a complex number are $\left(\frac{360}{n}\right)°$ apart, we need to ensure that $342 + \frac{360}{n} > 360$. (Note that it cannot be equal, since that would mean that z_2 is real.) This means that $\frac{360}{n} > 18$, which means $n < 20$, making the desired quantity 19.

5-2 If $z = \left(\frac{1+7i}{2+bi}\right)^4$ where b is a real number and z has the same modulus as $3.2 + 2.4i$, find b^2.
[Answer: 21]

Note that the modulus of $3.2 + 2.4i$, which is $(.8)(4 + 3i)$, is $(.8)(5) = 4$. This means that $4 = \left(\frac{\sqrt{50}}{\sqrt{4+b^2}}\right)^4 = \frac{2500}{(4+b^2)^2}$, so $(4 + b^2)^2 = 625$, and thus $4 + b^2 = 25$, making $b^2 = 21$.

5-3 If A is an angle such that $\sin\left(A + \frac{3\pi}{4}\right) = \frac{\sqrt{5}}{8}$, then $\sin(2A) = \frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 59]

Using the $\sin(A + B)$ formula, we have $\sin(A)\cos\left(\frac{3\pi}{4}\right) + \cos(A)\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{5}}{8}$. This means $\frac{\sqrt{2}}{2}(\cos(A) - \sin(A)) = \frac{\sqrt{5}}{8}$, and therefore $\cos(A) - \sin(A) = \frac{\sqrt{10}}{8}$. From here we can square both sides to get $1 - 2\sin(\mathrm{A})\cos(\mathrm{A}) = \frac{5}{32}$, which means $2\sin(A)\cos(A) = \sin(2A) = \frac{27}{32}$, making the desired quantity $27 + 32 = 59$.

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Round 6: Conic Sections

6-1 One of the foci of the ellipse $\frac{(x-3)^2}{50}+\frac{(y+7)^2}{14}=1$ has a positive x-coordinate a. What is the value of a?
[Answer: 9]

Since the number beneath the x term is larger, the foci lie on the x-axis, and are located $\sqrt{50-14}=6$ units away from the center, which is located at $(3,-7)$. This means the foci have x-coordinates of 9 and -3, making the desired quantity 9.

6-2 A hyperbola has an asymptote with an equation of $y=\frac{1}{2}x+\frac{7}{2}$ and a range of $(-\infty,2]\cup[6,\infty)$. The largest y-value of the hyperbola where $x=4$ is $\frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a+b$.
[Answer: 15]

Because the range is not all reals, the hyperbola must be vertically oriented. The center of the hyperbola must have a y-coordinate of 4, and the x-coordinate can be solved for using the asymptote equation: $4=\frac{1}{2}x+\frac{7}{2}\rightarrow x=1$. Knowing that the slope of the asymptote is $\frac{1}{2}$ and the distance from the center to a vertex is 2 gives the equation: $-\frac{(x-1)^2}{4^2}+\frac{(y-4)^2}{2^2}=1$. Substituting $x=4$ gives $-\frac{9}{16}+\frac{(y-4)^2}{4}=1$, or $(y-4)^2=\frac{25}{4}$, or $y=4\pm\frac{5}{2}$. The largest possible value is therefore $\frac{13}{2}$, making the desired quantity $13+2=15$.

6-3 An ellipse is centered at origin and has a focus at $(0,\sqrt{2})$ and an area of $4\sqrt{3}\pi$. A circle is also centered at the origin and intersects the ellipse at points that lie on $y=x$ and $y=-x$. The square of the radius of the circle is $\frac{m}{n}$ where m and n are positive integers with no common factors greater than 1. Find $m+n$. (Note: the area of an ellipse is πab where a and b are semi-major and semi-minor axis lengths.)
[Answer: 55].

The ellipse has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and from the coordinates of the focus we know $b^2-a^2=2$ and $ab=4\sqrt{3}$, so $a^2b^2=48$. Substituting $a^2=\frac{48}{b^2}$ yields the equation b^4-2b^2-

$48 = 0$, yielding $b^2 = 8$, and so $a^2 = 6$, making the equation $\frac{x^2}{6} + \frac{y^2}{8} = 1$. Letting $y = x$ yields $\frac{7x^2}{24} = 1$, so the x and y coordinates of the points of intersection with the circle satisfy $x^2 = y^2 = \frac{24}{7}$, and so the equation of the circle is $x^2 + y^2 = \frac{48}{7}$, making the desired quantity $48 + 7 = 55$.

FAIRFIELD COUNTY MATH LEAGUE 2024-2025
Match 4
Team Round

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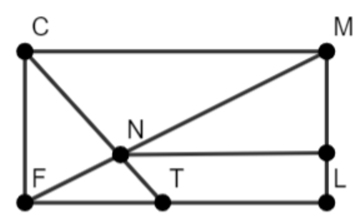
1. The geometric mean of a set of n numbers is the nth root of the product of the numbers. A set of three distinct positive integers greater than 1 has the property that its arithmetic and geometric means are both integers. What is the smallest possible value of the arithmetic mean of the set?
[Answer: 7]

The product of the numbers must be a perfect cube and the sum of the numbers must be a multiple of 3. An easy way to ensure the sum is a multiple of three is to have 3 be a factor of each element. The product will therefore contain 3^3 as a factor. Allowing one of the numbers to have a factor of 2 and another to have a factor of 2^2 will then make the product 6^3, and the sum will be a multiple of three. The desired quantity is therefore $\frac{3+2*3+2^2*3}{3} = 7$.

2. The quadratic equations $x^2 - 8x + p = 0$ and $x^2 - 2x + q = 0$, where p and q are real constants, each have two positive solutions for x. They share one solution and the other solutions are reciprocals of each other. If $p = a + b\sqrt{c}$, where $a, b,$ and c are positive integers and c has no perfect square factors greater than 1, find $a + b + c$.
[Answer: 17]

Letting the solutions to the first problem be m and n, we have $m + n = 8$, $mn = p$, and $m + \frac{1}{n} = 2$. Therefore $n - \frac{1}{n} = 6$, or $n^2 - 6n - 1 = 0$. This yields $n = 3 \pm \sqrt{10}$, but since n is positive, $n = 3 + \sqrt{10}$. This means $m = 8 - \left(3 + \sqrt{10}\right) = 5 - \sqrt{10}$ and $p = \left(3 + \sqrt{10}\right)\left(5 - \sqrt{10}\right) = 5 + 2\sqrt{10}$, making the desired quantity $6 + 35 + 12 = 53$.

3. Consider rectangle $FCML$, with $FC = 8$ and $CM = 10$. Point N is draw on diagonal $\overline{FM}$ such that the distance from N to $\overline{ML}$ is 7. Point T is drawn on $\overline{FL}$ such that $\overline{CT}$ contains point N. $LT = \frac{a}{b}$, where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 47]



See the diagram (not drawn to scale). We will call the point where the line segment showing the distance from N to $\overline{LM}$ intersects $\overline{LM}$ point O. Note triangle FLM is similar to triangle NOM. Therefore $\frac{10}{8} = \frac{7}{OM}$, so $OM = 5.6$ and $LO = 2.4$. Also, triangle CMN is similar to triangle TFN, so $\frac{10}{FT} = \frac{5.6}{2.4} = \frac{7}{3}$. This means $FT = \frac{30}{7}$, so $LT = 10 - \frac{30}{7} = \frac{40}{7}$, making the desired quantity $40 + 7 = 47$.

4. If y varies directly as the second power of x and the ordered triple (a, b, c) has the properties that $a, b,$ and c are all different positive integers and both (a, b) and (b, c) fit this particular variation relationship, find the smallest possible value of $a + b + c$ when $b = 30$.
[Answer: 165]

From the problem, $\frac{b}{a^2} = \frac{c}{b^2}$, so $b^3 = a^2c$. If $b = 30$, then $(2^3)(3^3)(5^3) = a^2c$. Factoring out possible perfect squares, we have $a^2 \in \{1^2, 2^2, 3^2, 5^2, 6^2, 10^2, 15^2\}$. Note that a^2 cannot equal 30^2 because then $a = 30$ and the three integers must be different. The smallest total sum will occur with the largest possible value of a (to ensure the smallest possible value of c), making the ordered triple $(15,30,120)$ and the desired quantity $15 + 30 + 120 = 165$.

5. If k is a positive number such that $\arctan\left(\frac{1}{3}\right) + \arctan(k) = \arctan\left(\frac{2}{3}\right)$, then $k = \frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 14]

Taking the tangent of both sides yields $\tan\left(\arctan\left(\frac{1}{3}\right) + \arctan(k)\right) = \frac{2}{3}$. Using the $\tan(A + B)$ formula yields $\frac{\frac{1}{3}+k}{1-\frac{k}{3}} = \frac{2}{3}$, which becomes $\frac{1}{3} + k = \frac{2}{3} - \frac{2}{9}k$, or $\frac{11}{9}k = \frac{1}{3}$, making $k = \frac{3}{11}$, and thus the desired quantity is $3 + 11 = 14$.

6. The circle $x^2 + y^2 - 16x - 6y + 48 = 0$ and the line $x = k$, which lies to the left of the center of the circle, intersect at points A and B such that $AB = 8$. A particular conic section represents the set of all points equidistant from the center of the circle and the line. Find the x-coordinate of the two intersection points between the conic section and the circle.
[Answer: 10]

The circle's equation can be put into center-radius form as $x^2 - 16x + 64 + y^2 - 6y + 9 = 25$, or $(x - 8)^2 + (y - 3)^2 = 25$. Since the radius of the circle is 8 and the length of half the chord $\overline{AB}$ is 4, the distance from the center of the circle to the chord is 3 by the Pythagorean theorem, making $k = 5$. The problem therefore describes a parabola with a vertex of (6.5,3) and with an equation of $x = \frac{1}{4\left(\frac{3}{2}\right)}(y - 3)^2 + 6.5$. Since $(y - 3)^2 = 25 - (x - 8)^2$, we have $x = \frac{1}{6}(25 - (x - 8)^2) + 6.5$. Expanding and multiplying by 6 yields $6x = -x^2 + 16x - 39 + 39$, or $x^2 - 10x = 0$. The result $x = 0$ is extraneous, so the correct answer is $x = 10$.