FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 3
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Decimals and Base Notation

1-1 The decimal $.20\overline{24}$, when written as a fraction in simplest terms, is $\frac{a}{b}$ where a and b are relatively prime integers. Find $a+b$.
[Answer: 992]

The decimal can be broken up as $.20 + .00\overline{24}$, which is written in fraction form as $\frac{20}{100} + \frac{1}{100}\left(\frac{24}{99}\right)$, which becomes $\frac{20(99)+24}{100(99)} = \frac{2004}{9900}$, which reduces to $\frac{167}{825}$, making the desired quantity $167 + 825 = 992$.

1-2 If x and y are positive integer bases such that $24_x + 63_y = 41_{x+y}$, then write the value of 10001_{x-y} as a numeral in base 10.
[Answer: 82]

Setting up an equation from the given information yields $2x + 4 + 6y + 3 = 4(x + y) + 1$, which simplifies to $x - y = 3$. Therefore the desired quantity is $3^4 + 1 = 82$.

1-3 Let m, n, and q be positive integers such that $(16^m)(15^n) = 8.1 * 10^q$. If m, n, and q are then used as digits in the base $(q+1)$ number mnq_{q+1}, express this number as a numeral in base 10.
[Answer: 65]

Rewriting the expression using prime bases, we have $(2^{4m})(3^n)(5^n) = 8.1 * 10^q$. From this we know that $4m = n$, since we have no powers of 2 or 5 in the decimal part of the result. We also know that $n = 4$ to yield 81, so consequently $m = 1$, and $q = 4 + 1 = 5$. Therefore we are writing the number 145_6 as a numeral in base 10, making the desired quantity $6^2 + 4(6) + 5 = 65$.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 3
Individual Section

Please write your answers on the answer sheet provided.

Round 2: Word Problems

2-1 Mr. Bearse spent $\frac{1}{3}$ of his holiday bonus check on a new pair of shoes. He then spent $\frac{1}{3}$ of the remaining money on a new briefcase. Finally he spent $\frac{3}{4}$ of what now remained of his bonus on a new pet lizard. If he now had $125 left over to buy gifts for friends, how much in dollars did the briefcase cost?
[Answer: 250]

Briefcase	Briefcase	Shoes
Lizard	Lizard	
Lizard	$125	

As a bar model solution, 1/3 of the check was shoes. 1/3 of the remaining amount becomes 2/9 of the total, and ¾ of the remaining after that becomes 3/9 of the total, leaving 1/9 of the total equaling $125. Since the briefcase is worth twice that amount, its cost in dollars is $2(125) = 250$.

2-2 Mr. Zucca is grading tests from three classes, each with the same number of tests. He grades each class on a different day: one on Monday, one on Tuesday, and one on Wednesday. His grading on Tuesday is twice as fast as his grading on Monday, and on Wednesday, he grades 50% faster than his grading on Tuesday. If it takes him a total of 2 hours and 12 minutes to grade all three classes' tests, how many minutes did he spend grading tests on Monday?
[Answer: 72]

Let t be the number of minutes spent grading Monday. On Tuesday the time spent grading is $\frac{t}{2}$, and on Wednesday the time spent grading is $\left(\frac{2}{3}\right)\frac{t}{2} = \frac{t}{3}$. This means $t + \frac{t}{2} + \frac{t}{3} = 132$, or $\frac{11}{6}t = 132$, making the desired quantity $\frac{6}{11}(132) = 6(12) = 72$.

2-3 During a 2-hour period from 9:00 AM to 11:00 AM at FCML world, Connecticut's mathiest amusement park, patrons enter at a rate of 20 people per minute and leave at a rate of 8 people per minute. At 9:00 AM, 70 patrons are in line for the Graphinator ride, and the line is increasing at a rate of 25% of the net change in the park's population. The rate of people joining the line increases additionally every 20 minutes by an additional 10% of the park's net change in population. The ride processes people at a rate of 5 patrons per minute. Let m and n be the smallest and largest number of people in line for the Graphinator during the 2-hour window, respectively. Find $m + n$.
[Answer: 204]

We first see that the net change in the park's population is $20 - 8 = 12$ people per minute. Initially there are 70 people in line. The line is increasing at a rate of 3 people per minute but decreasing at a rate of 5 people per minute, making a net change of -2 people per minute. Therefore at the end of 20 minutes there are $70 - 20(2) = 30$ patrons in line. For the next twenty minutes the line will change by a rate of $3 + 1.2 - 5 = -.8$ patrons per minute, so after 40 minutes there are $30 - 20(.8) = 14$ people in line. After this point the line will have a net increase, so after the next 4 intervals of 20 minutes, the number of people in line will be $14 + 20(.4) + 20(1.6) + 20(2.8) + 20(4.0) = 14 + 20(8.8) = 190$. Therefore the desired quantity is $14 + 190 = 204$.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 3
Individual Section

Please write your answers on the answer sheet provided.

Round 3: Polygons

3-1 For how many values of $n < 2024$ does an $n-$gon have an odd number of sides but a positive even number of diagonals?
[Answer: 505]

The equation for the number of diagonals of an $n-$gon is $\frac{n(n-3)}{2}$. If n is odd but this expression is even, $n-3$ must be a positive multiple of 4. This means that $n-3 \in \{4,8,12,\ldots,2020\}$, making desired quantity $\frac{2020}{4} = 505$.
-

3-2 A regular n-gon contains the adjacent vertices A, B, and C. If $m\angle B = 10m\angle BAC$, find the value of n.
[Answer: 12]

Because the polygon is regular, triangle ABC is isosceles with vertex angle B. Letting $m\angle BAC = x°$, we have $x + x + 10x = 180$, so $x = 15$. This means $m\angle B = 150°$, and therefore $180 - \frac{360}{n} = 150$, making $n = 12$.

3-3 A regular n-gon has the property that the number of its sides added to the measure of one interior angle measure in degrees is equal to the measure in degrees of one interior angle of an $\frac{8}{3}n$-gon. Find the value of n.
[Answer: 15]

Setting up an equation from the description yields $n + 180 - \frac{360}{n} = 180 - \frac{360}{\frac{8}{3}n}$, or $n - \frac{360}{n} = -\frac{135}{n}$, which yields $n^2 - 360 = -135$, or $n^2 = 225$, so $n = 15$.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 3
Individual Section

Please write your answers on the answer sheet provided.

Round 4: Function and Inverses

Note: the inverse f^{-1} of a function is not necessarily a function.

4-1 Let $f(x)$ be a linear function such that $f(6) = 26$ and $f^{-1}(6) = 1$. Find $f(10)$.
[Answer: 42]

Since $f(x)$ is a linear function that contains the points $(6,26)$ and $(1,6)$, we can find its equation to be $f(x) = 4x + 2$. This makes the desired quantity $f(10) = 4(10) + 2 = 42$.

4-2 Let $f(x) = \sqrt{x-7}$ and $g(x) = 3f(x) - 10$. There exist values a and b such that $f^{-1}(a) = g^{-1}(a) = b$. Find the value of b.
[Answer: 32]

Since $g\left(f^{-1}(a)\right) = a$, we have $3f\left(f^{-1}(a)\right) - 10 = a$, or $3a - 10 = a$, making $a = 5$. This means $f(b) = 5$, so $\sqrt{b-7} = 5$, making $b = 32$.

4-3 Consider the functions $f(x) = \frac{1}{x^2 - a}$ and $g(x) = \log_2(3x + b)$ where a and b are positive constants. If the domain of $f \circ g$ is $\left(-\frac{11}{3}, k\right) \cup \left(k, \frac{5}{3}\right) \cup \left(\frac{5}{3}, \infty\right)$, then $|k| = \frac{p}{q}$ where p and q are positive integers with no common factors greater than 1. Find $p + q$.
[Answer: 223]

We know the domain of g is $\left(-\frac{b}{3}, \infty\right)$, so the domain of $f \circ g$ tells us that $b = 11$. The values of k and $\frac{5}{3}$ come from points where the value of $f \circ g$ is undefined, which will occur when $g(x)^2 - a = 0$. Since $g\left(\frac{5}{3}\right) = 4$, we have $4^2 - a = 0$, so $a = 16$. This implies that $g(k) = -4$, so $2^{-4} = 3k + 11$. Solving for k yields $k = -\frac{175}{48}$, making the desired quantity $175 + 48 = 223$.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 3
Individual Section

Please write your answers on the answer sheet provided.

Round 5: Exponents & Logarithms

5-1 If $x, y,$ and z are integers such that $(12^x)(18^y) = 24^z$, find $-\frac{x}{y}$.
[Answer: 5]

Writing in terms of prime bases yields $(2^{2x})(3^x)(2^y)(3^{2y}) = (2^{3z})(3^z)$, or $(2^{2x+y})(3^{x+2y}) = (2^{3z})(3^z)$, yielding the equation $2x + y = 3(x + 2y)$, or $-x = 5y$, so $-\frac{x}{y} = 5$.

5-2 Consider the equation $\log_4(x^2 + x + 6) - \log_4(x - 3) = 2$. If the equation has exactly one real solution p, find the value of p^2. If the equation has two real solutions p and q, find the value of $p^2 + q^2$. If the equation has no real solutions, write your solution as 2024.
[Answer: 117]

Rewriting the equation without logarithms yields $\frac{x^2+x+6}{x-3} = 4^2$, which simplifies to $x^2 - 15x + 54 = 0$. This is factorable into $(x - 6)(x - 9) = 0$, yielding possible solutions of $x = 6$ and $x = 9$. Both solutions produce valid results in the original equation, making the desired quantity $6^2 + 9^2 = 117$.

5-3 If $u = \log_2(3)$ and $v = \log_3(2)$, then the expression $(\log_2(9) + \log_3(8))(\log_9(16) + \log_8(27))$ is equivalent to $au^2 + bv^2 + c$ where $a, b,$ and c are positive integers. Find $a + b + c$.
[Answer: 15]

Multiplying out the expression yields $\log_2(9)\log_9(16) + \log_2(9)\log_8(27) + \log_3(8)\log_9(16) + \log_3(8)\log_8(27)$. Noting that $\log_9(a) = \frac{1}{2}\log_3(a)$, $\log_8(a) = \frac{1}{3}\log_2(a)$, and $\log_a(b)\log_b(c) = \log_a(c)$, this expression becomes $\log_2(16) + (2\log_2(3))\left(\frac{1}{3} * 3\log_2(3)\right) + (3\log_3(2))\left(\frac{1}{2} * 4\log_3(2)\right) + \log_3(27)$, which simplifies to $2(\log_2(3))^2 + 6(\log_3(2))^2 + 7$, making the desired quantity $2 + 6 + 7 = 15$.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 3
Individual Section

Please write your answers on the answer sheet provided.

Round 6: Matrices

6-1 If $\begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$ has determinant k, find is the determinant of $\begin{bmatrix} k & 1 \\ 2 & 3 \end{bmatrix}$.
[Answer: 19]

From the first matrix we have $k = (5)(2) - (3)(1) = 7$, making the determinant of the second matrix $(7)(3) - (1)(2) = 19$.

6-2 If $A = \begin{bmatrix} 4 & 2 \\ 6 & 9 \end{bmatrix}$ and B is a $2x2$ matrix such that $AB = \begin{bmatrix} -2 & 4 \\ -9 & 6 \end{bmatrix}$, find the determinant of $A + B$.
[Answer: 21]

One way to approach this problem is to determine what B should be based on the transformation that occurs to A to produce AB. The numbers are preserved in magnitude, but the columns are switched and the first column is negated. The matrix $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ accomplishes this transformation, making $A + B = \begin{bmatrix} 4 & 3 \\ 5 & 9 \end{bmatrix}$, and thus making the desired quantity $(4)(9) - (3)(5) = 21$.

6-3 The matrix $A = \begin{bmatrix} x & y & x \\ 2 & 2 & 1 \\ 7 & 9 & 10 \end{bmatrix}$, where x and y are positive integers less than 100, has an inverse with a determinant of $\frac{1}{2}$, find the largest possible value of x.
[Answer: 79]

Since the determinant of A^{-1} is $\frac{1}{2}$, the determinant of A is 2. Therefore, finding the determinant of A and setting it equal to 2 yields $x\big((2)(10) - (1)(9)\big) - y\big((2)(10) - (1)(7)\big) + x\big((2)(9) - (2)(7)\big) = 2$, which simplifies to $15x - 13y = 2$. Note that (1,1) is an ordered pair solution to this equation. Setting up $1 + 15n < 100$ to find the largest value of y yields $n = 6$ as the largest possible value. Therefore the largest possible value of x is $1 + (6)(13) = 79$.

FAIRFIELD COUNTY MATH LEAGUE 2024-2025
Match 3
Team Round

Please write your answers on the answer sheet provided.

1. If x is an integer base such that $62_{3x} = 205_x + 45_{x+1}$, express the value of 2024_x as a numeral in base 10.
[Answer: 448]

Setting up an equation from provided information yields: $6(3x) + 2 = 2x^2 + 5 + 4(x + 1) + 5$, which simplifies to $2x^2 - 14x + 12 = 0$, or $x^2 - 7x + 6 = 0$, which is factorable into $(x - 6)(x - 1) = 0$. Since $x = 1$ cannot be a base of a numeral 205, the solution must be $x = 6$. This makes the desired quantity $2(6^3) + 2(6) + 4 = 432 + 12 + 4 = 448$.

2. For a fall banquet, Andrew and Mike cut out decorative paper leaves over three days and make the same total number of leaves each day. Their work rates are constant but not necessarily equal. On day one, they work together the entire time, but on day two, Mike arrives ten minutes late, causing the total time Andrew has to make leaves that day to increase by 25%. Frustrated, Andrew refuses to show up on day three, leaving Mike to make all the remaining alone. If Mike ended up making 198 of the 360 total leaves across the three days, how many leaves did Mike make the second day?
[Answer: 30]

Let a be Andrew's rate in leaves per minute and m be Mike's rate in leaves per minute. Since there are 360 total leaves, we know that they make 120 leaves on each of the three days. That also means that Mike made 78 leaves in the first two days. If t is the time spent making leaves the first day, we have $(a + m)t = 120$ and $a(1.25t) + m(1.25t - 10) = 120$. The second equation is writeable as $1.25(a + m)t - 10m = 120$, and we can substitute to get $1.25(120) - 10m = 120$, which can be solved to yield $m = 3$. Finally, we have $mt + 1.25mt - 10m = 78$, or $3t + 3.75t - 30 = 78$, or $6.75t = 108$, giving $t = 16$. Therefore, the desired quantity is $3(1.25(16) - 10) = 3(10) = 30$.

3. There are 12 ordered pairs $(m, n), n > m$, such that the difference in degrees of one interior angle of a regular n-gon and one interior angle of a regular m-gon is 10 degrees. The smallest possible measure of an exterior angle in degrees of one of the n-gons is $\frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 9]

Setting up $180 - \frac{360}{n} - \left(180 - \frac{360}{m}\right) = 10$, we simplify to $\frac{360}{m} - \frac{360}{n} = 10$. We can solve for n to yield $n = \frac{360m}{360=10m} = \frac{36m}{36-m}$. The largest value of n will occur when $m = 35$, making $n = 36(35)$. The measure of an exterior angle of this polygon is then $\frac{360}{36(35)} = \frac{10}{35} = \frac{2}{7}$, making the desired quantity $2 + 7 = 9$.

4. A function $f(x)$ has the property that for all $a > -1$, $f(a) = f^{-1}(2a+1)$. If $f(1) = 2$, find the value of $f(63)$.
[Answer: 95]

Setting $f(1) = 2 = f^{-1}(2 * 1 + 1) = f^{-1}(3)$, which means $f(2) = 3$. We now have $f(2) = 3 = f^{-1}(2 * 2 + 1) = f^{-1}(5)$, and since $f^{-1}(5) = 3$, we have $f(3) = 5$. From here we can establish a pattern of repeating x and y values, using the fact that every new y value is one more than twice the previous x value. Extrapolating as showing in the following table yields:

x	1	2	3	5	7	11	15	23	31	47	63
$f(x)$	2	3	5	7	11	15	23	31	47	63	95

Therefore the desired value is 95.

5. The equation $(\log_4(16x))\left(\log_4\left(\frac{x^2}{4}\right)\right) = \log_x(64)$ has three positive real solutions a, b, and c where $a < b < c$. Find $\frac{c}{ab}$.
[Answer: 128]

Using properties of logs, we can rewrite this equation as $(\log_4(x) + 2)(2\log_4(x) - 1) = 3\log_x(4)$. Then, noting that $\log_x(4) = \frac{1}{\log_4(x)}$ and distributing the products on the left, we have $2(\log_4(x))^2 + 3\log_4(x) - 2 = \frac{3}{\log_4(x)}$, and finally we can multiply every term by $\log_4(x)$ and subtract to produce $2(\log_4(x))^3 + 3(\log_4(x))^2 - 2\log_4(x) - 3 = 0$. Substituting $u = \log_4(x)$ yields $2u^3 + 3u^2 - 2u - 3 = 0$, which factors to make $(2u+3)(u+1)(u-1) = 0$. This means we have $x \in \left\{4^{-\frac{3}{2}}, 4^{-1}, 4\right\}$, and therefore $a = \frac{1}{8}$, $b = \frac{1}{4}$, and $c = 4$, making the desired quantity $4(8)(4) = 128$.

6. If the value of x is such that the matrices $A = \begin{bmatrix} 2x & 1 \\ -1 & 4x^2 \end{bmatrix}$ and $B = \begin{bmatrix} 2x+1 & -13 \\ 2x+1 & 11x \end{bmatrix}$ have equal determinants, find the largest possible value of the determinant of A.
[Answer: 513]

Setting up an equation by finding the determinants yields $8x^3 + 1 = (2x+1)(11x) - (-13)(2x+1)$. Factoring the left side yields $(2x+1)(4x^2 - 2x + 1) = (2x+1)(11x+13)$. This means one of the possible values of $x = -\frac{1}{2}$. Considering the remaining factors, we have $4x^2 - 2x + 1 = 11x + 13$, or $4x^2 - 13x - 12 = 0$, which factors into $(x-4)(4x+3) = 0$, producing the additional values $x = 4$ and $x = -\frac{3}{4}$. The largest value of the determinant of A will occur when $x = 4$, producing the desired quantity $8(4^3) + 1 = 513$.