FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 2
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Factors and Multiples

1-1 If $M = lcm(18,24)$ and $N = lcm(12,45)$, find $lcm(M, N)$.
[Answer: 360]

$M = \frac{18*24}{6} = 72$ and $N = \frac{12*45}{3} = 180$, so the desired quantity is $\frac{72*180}{36} = 360$.

1-2 The number n has exactly 60 factors, including 1 and itself, and the largest possible number of trailing zeros. What is the smallest possible value of n?
[Answer: 1600000]

To maximize the number of trailing zeros, we need to maximize the number of 2's and 5's in the prime factorization of the result. The prime factorization of 60 is $2^2 * 3 * 5$, we can create a number with at most 5 trailing zeros, using $60 = 10 * 6$ and making a number of the form a^9b^5. Let $a = 2$ and $b = 5$ to keep the result as small as possible. This makes $n = 2^4 * 10^5 = 1600000$, which is the desired quantity.

1-3 Let k be a positive integer such that $gcf(k, 54) = 18$ and $lcm(k, 540) = 3780$. If A is the sum of all possible values of k and B is the greatest common factor of all possible values of k, find $\frac{A}{B}$.
[Answer: 18]

Since $54 = 2 * 3^3$ and $18 = 2 * 3^2$, it follows that k has at least one factor of 2 and exactly 2 factors of 3. Because $540 = 2^2 * 3^3 * 5$ and $3780 = 2^2 * 3^3 * 5 * 7$, we know k has at most 2 factors 2 and at most 1 factor of 5. Additionally, k must have a factor of 7. Therefore, k can have one of the following factorizations: $\{2 * 3^2 * 7, 2^2 * 3^2 * 7, 2 * 3^2 * 5 * 7, 2^2 * 3^2 * 5 * 7\}$. This makes $B = 2 * 3^2 * 7$ and $A = 2 * 3^2 * 7 * (1 + 2 + 5 + 10)$, making the desired quantity $1 + 2 + 5 + 10 = 18$.

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Round 2: Polynomials and Factoring

2-1 If $f(x) = x^2 + 3x - 1$ and $g(x) = x^2 - (f(2) - 1)x - f(11)$, find the positive zero of $g(x)$.
[Answer: 17]

Note $f(2) = 9$ and $f(11) = 153$, so $g(x) = x^2 - 8x - 153$, which factors into $g(x) = (x - 17)(x + 9)$, making the desired quantity 17.

2-2 The polynomial $f(x) = x^3 + ax^2 + bx + 2024$ has three integer zeros, one of which is shared with the polynomial $g(x) = 3x^2 - 31x - 22$. Find the smallest possible positive value of a.
[Answer: 4]

Note that $g(x)$ is factorable into $g(x) = (x - 11)(3x + 2)$, so $f(x)$ must have 11 as one of its integer zeros. This mean we need positive integers p and q such that $f(x) = (x - p)(x + q)(x - 11)$ has a constant term of 2024 and $q - p - 11$ is a positive number that is as small and as possible. Since $11pq = 2024$, we know $pq = 184 = 2^3 * 23$. Let $q = 23$ and $p = 8$ to get $a = 23 - 8 - 11 = 4$, which is the desired quantity.

2-3 There exist positive integers $a, b,$ and c such that if the term $ax^b y^c$ is added to and subtracted from the polynomial $27x^9 + 576x^3y^6 + 512y^9$, the resulting polynomial is factorable as a difference of the cubes of two polynomials in x and y with integer coefficients. Find $a + b + c$.
[Answer: 225]

Note that $27x^9 = (3x^3)^3$ and $512y^9 = (8y^3)^3$, and $576x^3y^6 = 3(3x^3)(8y^3)^2$. So adding the $3(3x^3)^2(8y^3) = 216x^6y^3$ produces the binomial cube $(3x^3 + 8y^3)^3$. In addition, $216x^6y^3 = (6x^2y)^3$, so subtracting it from the polynomial produces the difference of cubes $(3x^3 + 8y^3)^3 - (6x^2y)^3$, which would be factorable into two polynomials in x and y with integer coefficients. Therefore $216x^6y^3$ is the desired term, making the final desired quantity $216 + 6 + 3 = 225$.

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Round 3: Area and Perimeter

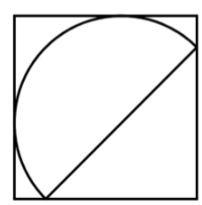
3-1 The difference between the area of a circle with radius k and a circle with diameter k is 1200π. Find the value of k.
[Answer: 40]

Setting up the equation $\pi k^2 - \pi\left(\frac{k}{2}\right)^2 = 1200\pi$, we can simplify to $\frac{3}{4}k^2 = 1200$, or $k^2 = 1600$, making the desired quantity $k = 40$.

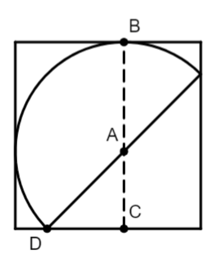
3-2 A "deathly hallows" is a symbol composed of an equilateral triangle ABC, an altitude $\overline{BD}$ with D on $\overline{AC}$, and a circle inscribed in the triangle. If Harry is going to use string to make a deathly hallows such that the inscribed circle has an area of 48π square inches, then the amount of string he would need in inches to make the whole figure is $a + b\sqrt{c} + d\sqrt{e}\pi$, where $a, b, c, d,$ and e are positive integers and c and e have no perfect square factors greater than 1. Find $a + b + c + d + e$.
[Answer: 98]

From the given information, the circle will have a radius of $4\sqrt{3}$, which is also one-third the length of the altitude of the triangle. Therefore the altitude length is $12\sqrt{3}$, giving the triangle a side length of 24. Therefore, the total length of the triangle perimeter, the altitude length, and the circumference of the circle is $72 + 12\sqrt{3} + 8\sqrt{3}\pi$, making the desired quantity $72 + 12 + 3 + 8 + 3 = 98$.

3-3 A semicircle with area 100π is inscribed in a square such that the diameter of the semicircle is parallel to a diagonal of the square, as shown in the figure. If the area of the square is A and the perimeter of the square is P, find $A - 5P$.
[Answer: 100]



See the diagram. From the given information, the radius of the circle must be $10\sqrt{2}$ (since the full circle would have an area of 200π). If A is the center of the circle, then $AB = 10\sqrt{2}$. Also note that triangle ACD is an isosceles right triangle with hypotenuse length $10\sqrt{2}$, so $AC = 10$. This means the square has side length $10 + 10\sqrt{2}$, and therefore $A = 300 + 200\sqrt{2}$ and $P = 40 + 40\sqrt{2}$. This makes the desired quantity $300 + 200\sqrt{2} - 200 - 200\sqrt{2} = 100$.



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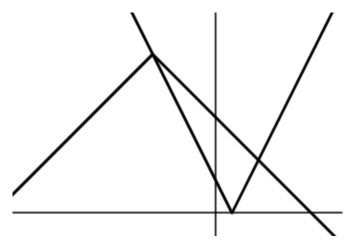
Round 4: Absolute Value & Inequalities

4-1 What is the largest integer that satisfies the inequality $|150x - 2024| < 1130$?
[Answer: 21]

Every quantity can be divided by 2 to make $|75x - 1012| < 565$. Therefore we are finding the largest integer value of x that satisfies $75x - 1012 < 565$, or $75x < 1577$. Note that $75(20) = 1500$, so $75(21) = 1575$, making 21 the desired result.

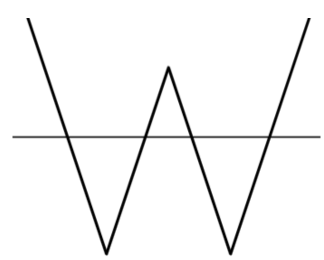
4-2 The functions $f(x) = |2x - 1|$ and $g(x) = 5 - |x + 2|$ share the ordered pairs (a, b) and (c, d). Find $|a| + |b| + |c| + |d|$.
[Answer: 10]

See the diagram. $f(x)$ has a vertex of $\left(0, \frac{1}{2}\right)$, opens upward, and has sides with slopes of 2 and -2. $g(x)$ has a vertex of $(-2,5)$, opens downward, and has sides of slopes 1 and -1. First note that $(-2,5)$ is a point on $f(x)$, so that I one of the intersection points. The other must be in quadrant one, being an intersection of $y = 2x - 1$ and $y = -(x + 2) + 5$. Setting $2x - 1 = -(x + 2) + 5$ yields $x = \frac{4}{3}$, making the y-coordinate $\frac{5}{3}$. This makes the desired quantity $2 + 5 + \frac{4}{3} + \frac{5}{3} = 10$.



4-3 The function $f(x) = \left||3x + 2024| - 6\right| - k$, where $0 < k < 6$, has four x-intercepts. If the area enclosed by $f(x)$ between the second and third x-intercepts and bounded below by the x-axis has an area of $\frac{27}{16}$, then $k = \frac{a}{b}$ where a and b are positive integers that have no common factors greater than 1. Find $a + b$.
[Answer: 19]

See the diagram. First note that the shift caused by the $+2024$ has no bearing on the result of the problem. To visualize this graph, first an absolute value graph with slides with slope 3 and -3 is shifted downward 6 units, followed by the part below the x-axis being reflected above the x-axis. This produces a “W” graph with a middle triangle above the x-axis with a height of 6 units. This whole graph is then shifted down k units, making a triangle between the second and third x-intercepts with a height of $6 - k$ units. Since the sides of the triangle have slopes of 3 and -3, then if the base of the triangle has length b, it follows that $\frac{6-k}{\frac{b}{2}} =$



3, and so $\frac{b}{2} = \frac{6-k}{3}$. Therefore $(6-k)\frac{6-k}{3} = \frac{27}{16}$, and $(6-k)^2 = \frac{81}{16}$, yielding $6 - k = \frac{9}{4}$ (the negative result produces a k outside the given range). This means $k = \frac{15}{4}$, making the desired quantity $15 + 4 = 19$.

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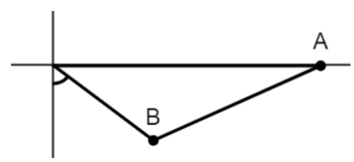
Round 5: Law of Sines and Cosines

5-1 In triangle ABC, $\sin(A) = \frac{3}{5}$, $\sin(B) = \frac{1}{15}$, and $AC + BC = 140$. Find BC.
[Answer: 126]

By the law of sines, $\frac{AC}{\sin(B)} = \frac{BC}{\sin(A)}$, or $\frac{\sin(A)}{\sin(B)} = \frac{BC}{AC} = 9$, so $BC = 9AC$. This mean $10AC = 140$ so $AC = 14$, and therefore $BC = 126$.

5-2 A hiker hikes 15 kilometers directly west from point A, then turns in a direction of $\alpha°$ East of South and hikes for 7 kilometers to point B. If at that point the hiker turns and walks directly back to point A, she will have to walk a distance of D kilometers from point B to point A. If $\cos(\alpha°) = \frac{3}{5}$, find D^2.
[Answer: 106]



See the diagram. Because the angle between the first two sides of the triangle is the complement of the triangle with measure $\alpha°$, this angle must have a cosine of $\frac{4}{5}$. Thefore by the law of cosines, $D^2 = 15^2 + 7^2 - 2(15)(7)\left(\frac{4}{5}\right) = 225 + 49 - 168 = 106$.

5-3 Consider triangle ABC with point D on $\overline{AC}$ such that $AD = DB$ and $CD = 2AD$. If $AB = 12$ and $\cos(\angle BDA) = \frac{1}{9}$, find BC.
[Answer: 21]

Let $AD = DB = x$. Then by the law of cosines, $12^2 = x^2 + x^2 - 2(x)(x)\left(\frac{1}{9}\right)$, so $\frac{16}{9}x^2 = 144$, making $x^2 = 91$, or $x = 9$. Therefore $CD = 18$, and $\cos(\angle BCD) = -\frac{1}{9}$, so $(BC)^2 = 9^2 + 18^2 - 2(9)(18)\left(-\frac{1}{9}\right) = 441$, so $BC = 21$.

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Round 6: Equations of Lines

6-1 The line $y = 5x + b$, where b is constant, passes through $(3,4)$. What is slope of a line that passes through $(-8,1)$ and $(b, 10b)$?
[Answer: 37]

Setting $4 = 5(3) + b$ yields $b = -11$. Then the slope between $(-8,1)$ and $(-11, -110)$ is $\frac{1-(-110)}{-8-(-11)} = \frac{111}{3} = 37$.

6-2 The line $3x - 5y = -10$ is reflected about the line $y = x$ to produce the function $g(x)$, and the two lines intersect at point P. The linear function $h(x)$ is perpendicular to $g(x)$ and intersects $g(x)$ at P. Find $h(-10)$.
[Answer: 14]

Reflecting a line across $y = x$ creates a function with the x and y coordinates from the original function reversed, and the two functions intersect at a point on $y = x$. This point can be found setting $3x - 5x = -10$, so $x = 5$ and $P = (5,5)$. The reflected line therefore has the equation $3y - 5x = -10$, and so $g(x)$ has a slope of $\frac{5}{3}$. Therefore $h(x) = -\frac{3}{5}(x - 5) + 5$, and $h(-10) = -\frac{3}{5}(-10 - 5) + 5 = 14$.

6-3 Two linear functions, $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$, intersect at (3,6). If $m_1 > 0$, $m_2 = -\frac{3}{2}m_1$, and the triangle formed by $f(x)$, $g(x)$, and the x-axis has an area of 37.5 square units, then $b_2 = \frac{p}{q}$ where p and q are positive integers with no common factors greater than 1. Find $p + q$.
[Answer: 53]

Let x_1 be the length of the base of the triangle to the left of $x = 3$ and let x_2 be the length of the base to the right of $x = 3$. Then $\frac{6}{x_1} = m_1$ and $m_2 = -\frac{6}{x_2} = -\frac{3}{2}m_1 = -\frac{3}{2}\left(\frac{6}{x_1}\right)$. This yields $x_1 = \frac{3}{2}x_2$ This means $\frac{1}{2}(6)\left(x_2 + \frac{3}{2}x_2\right) = 37.5$, or $\frac{5}{2}x_2 = 12.5$, making $x_2 = 5$. This means $g(x) = -\frac{6}{5}x + b_2$, and since $g(x)$ contains the point (3,6), we have $6 = -\frac{6}{5}(3) + b_2$, making $b_2 = \frac{48}{5}$, and therefore the desired value is $48 + 5 = 53$.

FAIRFIELD COUNTY MATH LEAGUE 2024-2025
Match 2
Team Round

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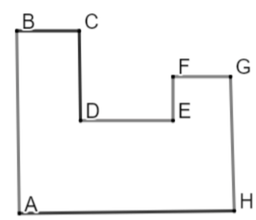
1-1 Let the set P be the set of the first 10 prime numbers. Let the set N be a set of ten positive integers with the property that for every element p in P, there is an element n in N such that the element n is the smallest positive number with exactly p factors. Find the median of the elements of N.
[Answer: 2560]

Note that $P = \{2,3,5,7,11,13,17,19,23,29\}$. For every element p in P, there is an element 2^{p-1} in N. That means the median of the element of N is the arithmetic mean of 2^{11-1} and 2^{13-1}, or $\frac{1}{2}(1024 + 4096) = \frac{1}{2}(5120) = 2560$.

1-2 What is the largest positive integer value $c < 10000$ such that the quadratic $x^2 - 2025x + c$ is factorable into two binomials with integer coefficients?
[Answer: 8084]

Note that c must be the product of two numbers that add to 2025. We see that $5 * 2020 = 10100$, which is greater than 10000, so testing $4 * 2021$, we get 8084, which is the desired quantity.

1-3 In the figure shown (not necessarily drawn to scale) all angles are right angles. $BC = 3$, $FG = 2$, $GH = 10$, and $CD = DE$. If the figure has a perimeter of 66 units and an area of 127 square units, then $EF = \frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 13]



Let $CD = DE = x$ and $EF = y$. This makes $AB = 10 + x - y$ and $AH = x + 5$.
The perimeter is therefore $10 + x - y + 3 + x + x + y + 2 + 10 + x + 5 = 30 + 4x$, so $30 + 4x = 66$ and therefore $x = 9$. The area is therefore $(14)(19 - y) - 9^2 - 2(9 - y) = 266 - 14y - 81 - 18 + 2y$, or $167 - 12y$. Setting this equal to 127 yields $y = \frac{40}{12} = \frac{10}{3}$, making the desired quantity $10 + 3 = 13$.

1-4 The inequality $a \le |x - b| \le c$, where $a, b,$ and c are real numbers, has the solution set $[x_1, x_2] \cup [x_3, x_4]$. The inequality $|2y - 5| \le 7$ has solution set $[y_1, y_2]$. If $y_1 + x_2 + x_4 = x_1 + x_3 + y_2$, $x_1 = -\frac{10}{3}$, and $x_3 = \frac{9}{2}$, then $b = \frac{p}{q}$ where p and q are positive integers with no common factors greater than 1. Find $p + q$.
[Answer: 10]

Rearranging the equation yields $y_2 - y_1 = x_2 - x_1 + x_4 - x_3$. Since y_1 and y_2 are the endpoints of the solution set to $\left|y - \frac{5}{2}\right| \le \frac{7}{2}$, it follows that $y_2 - y_1 = 7$. Therefore $x_2 - x_1 = x_4 - x_3 = \frac{7}{2}$, and $x_2 = -\frac{10}{3} + \frac{7}{2} = \frac{1}{6}$. This makes $b = \frac{1}{2}\left(\frac{1}{6} + \frac{9}{2}\right) = \frac{1}{2}\left(\frac{14}{3}\right) = \frac{7}{3}$, making the desired quantity $7 + 3 = 10$.

1-5 Consider triangles ABC and DEF which have equal areas. $AB < AC < DE < DF$, and the four values form an arithmetic sequence. If $\frac{\sin(D)}{\sin(A)} = \frac{2}{5}$, AB is an integer, and $DF < 2024$, find the largest possible value of AB.
[Answer: 1011]

Let $AB = x$. This makes $AC = x + a$, $DE = x + 2a$, and $DF = x + 3a$. We then have $\frac{1}{2}(x)(x + a)\sin(A) = \frac{1}{2}(x + 2a)(x + 3a)\sin(D)$. Then $\frac{\sin(D)}{\sin(A)} = \frac{x(x+a)}{(x+2a)(x+3a)} = \frac{2}{5}$. This means $5x^2 + 5ax = 2x^2 + 10ax + 12a^2$, or $3x^2 - 5ax - 12a^2 = 0$. This factors into $(x - 3a)(3x + 4a) = 0$. The solution $x = -\frac{4}{3}a$ is extraneous as it means $AB < 0$. Therefore $x = 3a$ and $DF = 2x$. Setting $2x < 2024$ means $x < 1012$, and the largest possible value of x is 1011.

1-6 A linear function $f(x)$ passes through the center of a circle with equation $(x - 8)^2 + (y - 21)^2 = 289$ and intersects the circle at its y-intercept. If $f(x)$ has an x −intercept of $(a, 0)$ where $a > 0$, then $a = \frac{p}{q}$ where p and q are positive integers with no common factors greater than 1. Find $p + q$.
[Answer: 101]

The radius of the circle is 17, and the line $f(x)$ intersects the circle at a point $(0, k)$. Since the center of the circle is $(8{,}21)$, then k must satisfy the equation $8^2 + (21 - k)^2 = 17^2$. This means $(21 - k)^2 = 15^2$, so $k = 6$ or $k = 36$. The value $k = 6$ is extraneous since $f(x)$ would have a positive slope and its x-intercept would be negative. Therefore $f(x)$ contains the points $(8{,}21)$ and $(0{,}36)$ and has the equation $g(x) = -\frac{15}{8}(x - 8) + 21$. Solving $-\frac{15}{8}(a - 8) + 21 = 0$ yields $a = \frac{96}{5}$, making the desired quantity $96 + 5 = 101$.