FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 1
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Percentages

1-1 A number x is decreased by 20% to make the number y. Then y is increased by $66\frac{2}{3}\%$ of 225% of itself. If this produces a final value of 2024, what is the value of x?
[Answer: 1012]

From the start we have $y = .8x$, and then $y + \left(\frac{2}{3}\right)\frac{9}{4}y = y + \frac{3}{2}y = \frac{5}{2}y = 2024$. Since $y = .8x$, $\frac{5}{2}(.8x) = 2x = 2024$, so $x = 1012$.

1-2 Crazy Jean's Doinkatorium is having a clearance sale: buy two doinks, get the more expensive one for 40% off and the less expensive one for 60% off. Marius finds a green doink he wants which is marked \$80, but is torn for his second doink between a spotted one which is $\$a$ and a striped one which is $\$b$. Marius notices he would save twice as much total money if he chose the spotted doink. If a and b are integers such that $b < 80 < a$, find the smallest possible value of b.
[Answer: 14]

From the problem, we have $.4a + .6(80) = 2(.4(80) + .6b)$, or $.4a + 48 = 64 + 1.2b$, which simplifies to $.4a - 1.2b = 16$, or $a - 3b = 40$. This gives $a = 3b + 40 > 80$, so the smallest possible value of b is 14.

1-3 Consider the rational number $p = \frac{20}{d}$, where d is a positive integer greater than 20 and less than 100. Increasing the 20 in numerator of p by $d\%$ and the d in the denominator of p by 20% increases the value of p by $n\%$ where n is a positive integer. Find the sum of the smallest and largest possible values of d.
[Answer: 124]

Setting up $\frac{20+\frac{d}{100}(20)}{d+\frac{20}{100}(d)} = \left(1 + \frac{n}{100}\right)\frac{20}{d}$, we can simplify the expression on the left to $\frac{20+\frac{d}{5}}{\frac{6}{5}d} = \frac{100+d}{6d}$. Multiplying both sides by $6d$ yields $100 + d = 120 + \frac{6}{5}n$. Rearranging this equation gives $5d - 6n = 100$. The lowest value of d that produces a positive integer value of n is $d = 26$, giving $(26,5)$ as an ordered pair. Continuing to increment d by 6 until reaching the largest value less than 100 yields $d = 98$ as part of the ordered pair $(98,65)$, making the desired quantity $26 + 98 = 124$.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 1
Individual Section

Please write your answers on the answer sheet provided.

Round 2: Solving Equations

2-1 Solve for x: $\sqrt{1+3\left(2-\left(5-4(1+2x)\right)\right)}=10$.
[Answer: 4]

Squaring both sides and simplifying the left hand side yields $4+24x=100$, which is solved to yield $x=4$.

2-2 The equation $\frac{1}{x}+\frac{2}{3}=m-8$, where m is a constant, has no solutions for x when $m=p$ and a solution of $x=\frac{3}{44}$ when $m=q$. Find $p+q$.
[Answer: 32]

Multiplying each term by $3x$ yields $3+2x=3x(m-8)$. This has no solutions if $m-8=\frac{2}{3}$, so $p=\frac{26}{3}$. Letting $x=\frac{3}{44}$ yields $3+\frac{6}{44}=\frac{9}{44}(q-8)$, or $132+6=9(q-8)$, yielding $q=8+\frac{138}{9}=\frac{70}{3}$. This makes the desired quantity $\frac{26}{3}+\frac{70}{3}=\frac{96}{3}=32$.

2-3 If a and b are positive constants such that the equation $ax+21=b(3x+a)$ has infinite solutions for x, find $(a+b)^2$.
[Answer: 112]

Setting $ax=3bx$, making $a=3b$, and $ab=21$ gives a system that can be solved multiple ways. One method is to set $(3b)b=3b^2=21$, so $b^2=7$ and $a^2=9b^2=63$, making $(a+b)^2=a^2+2ab+b^2=63+2(21)+7=112$.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 1
Individual Section

Please write your answers on the answer sheet provided.

Round 3: Triangles and Quadrilaterals

3-1 An equilateral triangle has a perimeter of k centimeters and an interior angle measure of $(3k-21)°$. What is the length of one side of the triangle in centimeters?
[Answer: 9]

Setting up $3k-21=60$ yields $k=27$, making the desired quantity $\frac{27}{3}=9$.

3-2 If an equilateral triangle has the same perimeter as an isosceles right triangle with area 18, then the area of the equilateral triangle is $a\sqrt{b}+c\sqrt{d}$ where $a, b, c,$ and d are positive integers and b and d have no perfect square factors greater than 1. Find $a+b+c+d$.
[Answer: 19]

An isosceles right triangle with legs of length x would have area $\frac{1}{2}x^2$, so setting this equal to 18 yields $x=6$. The perimeter of this isosceles right triangle would be $6+6+6\sqrt{2}=12+6\sqrt{2}$, making each side of the equilateral triangle of length $4+2\sqrt{2}$. Since the area of an equilateral triangle with side length l is $\left(\frac{l}{2}\right)\left(\frac{l}{2}\sqrt{3}\right)$, this makes the area $\left(2+\sqrt{2}\right)\left(2+\sqrt{2}\right)\left(\sqrt{3}\right)=6\sqrt{3}+4\sqrt{6}$, making the desired quantity $6+3+4+6=19$.

3-3 Consider kite $ABCD$ where $AB=BC=30$ and $m\angle A=m\angle D=m\angle C$. If the difference between the measures of the largest angle in the kite and the smallest angle in the kite is 40°, then the sum of all possible values of AC is $a+b\sqrt{c}$ where a, b, and c are positive integers and c has no perfect square factors greater than 1. Find $a+b+c$.
[Answer: 63]

If the angle measures in degrees are $x, x, x,$ and $x-40$, this means $4x-40=360$, making three of the angles have measures of $100°$ and $m\angle B=60°$. This means that $m\angle ABD=m\angle DBC=30°$, and so $AC=2(15)=30$. If the angle measures are $x, x, x,$ and $x+40$, this means that $4x+40=360$, making three of the angles have measures of $80°$ and $m\angle B=120°$. This means that $m\angle ABD=m\angle DBC=60°$, and so $AC=2\left(15\sqrt{3}\right)=30\sqrt{3}$. This means the sum of the two possible lengths is $30+30\sqrt{3}$, making the desired quantity $30+30+3=63$.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 1
Individual Section

Please write your answers on the answer sheet provided.

Round 4: Systems of Equations

4-1 If the ordered pair (a, b) solves the system $\begin{cases} 4x + 6y = 51 \\ y = 5x \end{cases}$, find $a + b$.
[Answer: 9]

Substituting $y = 5x$ into the first equation yields $4x + 6(5x) = 34x = 51$, making $x = \frac{3}{2}$. Consequently $y = 5\left(\frac{3}{2}\right) = \frac{15}{2}$, making the desired quantity $\frac{3}{2} + \frac{15}{2} = 9$.

4-2 If the system $\begin{cases} ax + by = 18 \\ 7x - 3y = a - 5 \end{cases}$ where a and b are constants has infinite solutions for (x, y) and $b > 0$, then $b = \frac{p}{q}$ where p and q are positive integers with no common factors greater than 1. Find $p + q$.
[Answer: 34]

Since the system must have infinite solutions, the equation $\frac{a}{7} = \frac{18}{a-5}$ must hold. This produces the equation $a^2 - 5a - 126 = 0$, which has solutions of $a = 14$ and $a = -9$. The value $a = 14$ would mean $b = -6$, but since $b > 0$, we have it that $a = -9$. Therefore $\frac{-9}{7} = \frac{b}{-3}$, making $b = \frac{27}{7}$, and thus the desired quantity is $27 + 7 = 34$.

4-3 The system $\begin{cases} \frac{5}{x+y} + \frac{3}{x-y} = \frac{x+y}{x-y} \\ 4x - y = A \end{cases}$, where A is a constant, has solutions (x_1, y_1) and (x_2, y_2) where $x_1 > x_2$. If $x_1 + x_2 = 12$, then $y_1 = \frac{a\sqrt{b}-c}{d}$ where $a, c,$ and d are relatively prime positive integers and b is a positive integer with no perfect square factors greater than 1. Find $a + b + c + d$.
[Answer: 58]

Combining the rational terms on the left side of the first equation yields $\frac{8x-2y}{(x+y)(x-y)} = \frac{x+y}{x-y}$, or $8x - 2y = (x + y)^2$. We can substitute from the second equation to get $2A = (x + y)^2$, which means $x + y = \pm\sqrt{2A}$. Adding this to the second equation yields $5x = A \pm \sqrt{2A}$, so the two values of x that solve the system are $x_1 = \frac{A+\sqrt{2A}}{5}$ and $x_2 = \frac{A-\sqrt{2A}}{5}$. The sum of these two values is $\frac{2}{5}A = 12$, so $A = 30$. This makes $x_1 = \frac{30+\sqrt{60}}{5}$, and since $y_1 = 4x_1 - 30$, we have $y_1 = \frac{120+4\sqrt{60}-150}{5} = \frac{8\sqrt{15}-30}{5}$, and thus our desired quantity is $8 + 15 + 30 + 5 = 58$.

FAIRFIELD COUNTY MATH LEAGUE 2023–2024
Match 1
Individual Section

Please write your answers on the answer sheet provided.

Round 5: Right Triangles

5-1 A spot on flat ground 2024 feet from the base of a skyscraper has an angle of elevation to the top of the skyscraper with a tangent of .75. What is the distance in feet from the spot on the ground to the top of the skyscraper?
[Answer: 2530]

Since the tangent of the angle is $\frac{3}{4} = \frac{\text{opposite}}{\text{adjacent}}$, then the cosine of the angle is $\frac{4}{5} = \frac{2024}{x}$. This makes the desired quantity $\frac{5}{4}(2024) = 2530$.

5-2 Right triangle TRI has right angle R. If TI and RI are integers that are 5 units apart and $0 < \cot(T) < 1$, find the smallest possible value of TI.
[Answer: 18]

This could be solved by inspection by testing possible hypotenuse lengths. For example, letting $TI = 10$ makes $RI = 5$ and $TR = \sqrt{75}$ which is greater than 5, making $\cot(T) > 1$. Continuing to increase the value of TI leads to the smallest possible value of 18.

More systematically, let $TI = a$, $RT = b$, and $RI = a - 5$. Then $b^2 + (a-5)^2 = a^2$, so $b^2 = 10a - 25$. Then $\cot(T) = \frac{RT}{RI} < 1$, so $b^2 < (a-5)^2$. Therefore $10a - 25 < (a-5)^2$, which yields $0 < a^2 - 20a + 50$, or $50 < (a-10)^2$. Therefore the smallest possible integer value of a is 18.

5-3 Consider right triangle ABC with right angle B and point D on $\overline{AC}$ and point E be on $\overline{AB}$ such that $\overline{BC}||\overline{DE}$. If $\tan(\angle CAB) = \frac{3}{4}\tan(\angle DBA)$ and $\cos(\angle DBA) = \frac{2}{5}$, find the least possible integer value of AB such $(BC)^2$ is an integer.
[Answer: 8]

We have $\frac{DE}{AE} = \frac{3DE}{4BE}$, so $BE = \frac{3}{4}AE$. Let $BE = 3k$ and $AE = 4k$. We also have $\frac{BE}{BD} = \frac{2}{5}$, so $BD = \frac{15}{2}k$. By Pythagorean theorem, $DE = \sqrt{\left(\frac{15}{2}k\right)^2 - (3k)^2} = \frac{3}{2}\sqrt{21}k$. Then by similarity $\frac{BC}{AB} = \frac{DE}{AE}$, so $BC = \frac{21\sqrt{21}}{8}k$. Now substituting $k = \frac{AB}{7}$, we have $BC = \frac{3\sqrt{21}}{8}AB$. Therefore, the least possible integer value of AB such that $(BC)^2$ is an integer is 8.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 1
Individual Section

Please write your answers on the answer sheet provided.

Round 6: Coordinate Geometry

6-1 If the graph of $f(x)$ is the perpendicular bisector of a line segment with endpoints $(1,6)$ and $(2,3)$, what is $f(27)$?
[Answer: 13]

The segment has slope -3 and midpoint $\left(\frac{3}{2},\frac{9}{2}\right)$, making the equation of the perpendicular bisector $f(x) = \frac{1}{3}\left(x-\frac{3}{2}\right)+\frac{9}{2}$. Therefore $f(27)=\frac{1}{3}\left(27-\frac{3}{2}\right)+\frac{9}{2}=\frac{1}{3}\left(\frac{51}{2}\right)+\frac{9}{2}=13$.

6-2 Point A has coordinates (j,k), and Point A is rotated 90° counterclockwise to make point B. If the midpoint of A and B is $\left(\frac{\sqrt{3}}{2},5\sqrt{3}\right)$, find the value of j^2-k^2.
[Answer: 30]

The coordinates of B will be $(-k,j)$. The midpoint of A and B would be $\left(\frac{j-k}{2},\frac{j+k}{2}\right)$. This means that $j-k=\sqrt{3}$ and $j+k=10\sqrt{3}$, making the desired quantity $(j+k)(j-k)=\left(\sqrt{3}\right)\left(10\sqrt{3}\right)=30$.

6-3 Circles with equations $(x-5)^2+(y-9)^2=9$ and $(x+1)^2+(y-1)^2=64$ intersect at points P and Q. $PQ=\frac{a\sqrt{b}}{c}$ where a and c are positive integers with no common factors greater than 1 and b is a positive integer with no perfect square factors greater than 1. Find $a+b+c$.
[Answer: 12]

The distance between the centers of the circles is 10 units, so one way to solve this is to let k be equal to half the distance PQ, then solving $\sqrt{64-k^2}+\sqrt{9-k^2}=10$. Subtracting the second radical and squaring both sides yields $64-k^2=100-20\sqrt{9-k^2}+9-k^2$, simplifying to $-45=-20\sqrt{9-k^2}$. This yields $9-k^2=\frac{81}{16}$, yielding $k^2=\frac{63}{16}$, making $k=\frac{3\sqrt{7}}{4}$. Finally $PQ=2k=\frac{3\sqrt{7}}{2}$, making the desired quantity $3+7+2=12$.

FAIRFIELD COUNTY MATH LEAGUE 2024–2025
Match 1
Team Round

Please write your answers on the answer sheet provided.

1. The positive integer k has the properties that reducing k by 20% produces an even integer, reducing k by 12.5% produces an odd integer, and while k is not a multiple of 9, increasing k by k% does produce a multiple of 9. Find the least possible value of k.
[Answer: 440]

We know k is divisible by 5 by the first descriptor, and we also know k is divisible by 8 with no other even factors from the second descriptor. This means $k = 40n$ where n is an odd integer but not a multiple of 9. Increasing $40n$ by $(40n)$% yields $40n\left(1 + \frac{40n}{100}\right) = 8n(5 + 2n)$. This is a multiple of 9 when $n = 2$, but n cannot be odd, so the first odd value of n that produces a multiple of 9 (where n is not a multiple of 9) is $n = 11$ to make $(88)(27)$. This makes the desired value of $k = 40(11) = 440$.

2. How many ordered pairs (x, y), where x and y are positive integers less than 100, solve the equation $5 + \frac{3y-42}{x-2y} = 2 - \frac{2x}{x-2y}$?
[Answer: 19]

First note that $x \neq 2y$ due to the expression in the denominator. Multiplying every term by $x - 2y$ yields $5x - 10y + 3y - 42 = 2x - 4y - 2x$, or $5x - 3y = 42$. Note that (9,1) is a valid ordered pair. Incrementing x by 3 and y by 5 yields (12,6), but this contradicts the domain restriction of $x \neq 2y$, so that solution is extraneous. The value of y can be further incremented by 5 a total of 18 times before exceeding 100, making a total of 19 ordered pairs.

3. Consider parallelogram $FCML$, with $FC = ML = 10$. The altitude from vertex F intersects $\overline{LM}$ at point P and the altitude from vertex M intersects $\overline{FL}$ at point Q. If the parallelogram has an area of 50 and $MQ = \frac{4}{3}FL$, then $(LP)^2 = \frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 27]

Since the area of the parallelogram is $(FP)(ML)$, we have $FP = 5$. Also, since the area of the parallelogram is $(MQ)(FL)$, we have $\frac{4}{3}(FL)^2 = 50$, making $(FL)^2 = \frac{75}{2}$. Since $(FL)^2 = (LP)^2 + (FP)^2$, we have $(LP)^2 = \frac{75}{2} - 25 = \frac{25}{2}$, making the desired quantity $25 + 2 = 27$.

4. If the ordered pair (a, b) solves the system $\begin{cases} \frac{2}{x} + \frac{4}{y} = 27 \\ 3x + 6y = 10 \end{cases}$, find the value of $\frac{a}{b} + \frac{b}{a}$.
[Answer: 20]

Multiplying the corresponding sides of the system together yields $6 + \frac{12x}{y} + \frac{12y}{x} + 24 = 270$. Therefore $30 + 12\left(\frac{x}{y} + \frac{y}{x}\right) = 270$, and therefore $\frac{x}{y} + \frac{y}{x} = 20$.

5. A right triangle with area 12 has legs whose lengths sum to 13. The length of the altitude from the vertex of the right angle to the hypotenuse is $\frac{p}{q}$ where p and q are positive integers with no common factors greater than 1. Find $p+q$.
[Answer: 35]

Let the lengths of the triangle be a and b. Then $a+b=13$, so $a^2+2ab+b^2=169$. Since the area of the triangle is $\frac{1}{2}ab=12$, this means that $2ab=48$, so $a^2+b^2=121$, so the hypotenuse has length 11. If x is the length of the altitude from the vertex to the hypotenuse, then $\frac{1}{2}(11)(x)=12$, so $x=\frac{24}{11}$, making the desired quantity $24+11=35$.

6. Point P on the line $y=5x$ is reflected across $y=x$ to a point P' on the line $y=\frac{1}{5}x$. If the distance from P to P' is 8 units, find the square of the distance from the origin to P.
[Answer: 52]

Consider the point $A=(1,5)$ on the line $y=5x$. The reflection of this point over $y=x$ is $A'=(5,1)$. The distance between A and A' is $\sqrt{(1-5)^2+(5-1)^2}=\sqrt{32}=4\sqrt{2}$, and the distance from the origin to A is $\sqrt{26}$. The triangle formed by the origin O, A, and A' is similar to the triangle formed by O, P, and P'. Therefore we can set up $\frac{OA}{AA'}=\frac{OP}{PP'}$, so $\frac{\sqrt{26}}{4\sqrt{2}}=\frac{OP}{8}$. This makes $OP=\sqrt{52}$, making the desired quantity $(OP)^2=52$.