FAIRFIELD COUNTY MATH LEAGUE 2023–2024
Match 6
Individual Section

Please write your answers on the answer sheet provided.

<u>Round 1: Lines and Angles</u>

1-1 An isosceles triangle has two angles whose measures differ by 15°. What is the measure in degrees of the supplement of the largest possible angle in the triangle?
[Answer: 110]

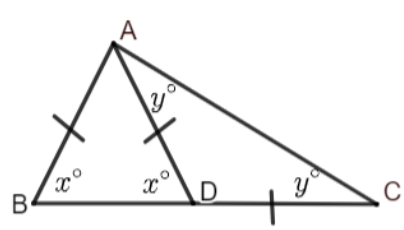
If the largest angle measure x belongs to the base angle, then $x + x + x - 15 = 180$, giving $x = 65$. If the largest angle measure belongs to the vertex angle, then $x + 2(x - 15) = 180$, giving $x = 70$. Since $70 > 65$, the desired quantity is $180 - 70 = 110$.

1-2 A convex decagon has one set of three congruent angles and another set of seven congruent angles. If all the angle measures in degrees are integers, what is the smallest possible measure in degrees of one angle of the decagon?
[Answer: 67]

The sum of all angle measures in a decagon is $(8)(180) = 1440$. Therefore we have $3a + 7b = 1440$, where a and b are positive integers less than 180. Note that $(11, 201)$ is a solution, and increasing a by 7 eight times and decreasing b by 3 eight times yields the valid ordered pair $(67{,}177)$, giving us our smallest possible angle measure in degrees, 67.

1-3 Consider triangle ABC with point D on $\overline{BC}$ such that $AB = AD = DC$. If $m\angle DAB > m\angle CAD$, find the largest possible integer value of $m\angle ABC$ in degrees.
[Answer: 71]

Refer to the diagram. We see that $x = 2y$, or $y = \frac{1}{2}x$. We then have $180 - 2x > \frac{1}{2}x$, which gives $x < 72$. Therefore the largest possible integer value of $m\angle ABC$ in degrees is 71.



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Round 2: Literal Equations

2-1 If $x = 10 + \frac{3}{w}$ and $y = 7x + 5$, then $w = \frac{a}{y-b}$ for some integers a and b. Find $a + b$.
[Answer: 96]

Since $x = \frac{y-5}{7}$, we have $\frac{y-5}{7} = 10 + \frac{3}{w}$, or $\frac{y-75}{7} = \frac{3}{w}$, or $\frac{w}{3} = \frac{7}{y-75}$. Finally $w = \frac{21}{y-75}$, making the desired quantity $21 + 75 = 96$.

2-2 The equation $x^2 + 3y = \sqrt{x^4 + 5x^2y + 8y^2 + \frac{416}{81}}$ has as one of its solutions $\left(\frac{a}{b}, \frac{4}{9}\right)$ where a and b are positive coprime integers. Find $a + b$.
[Answer: 13]

Squaring both sides yields $x^4 + 6x^2y + 9y^2 = x^4 + 5x^2y + 8y^2 + \frac{416}{81}$, which simplifies to $x^2y + y^2 = \frac{416}{81}$. Substituting $y = \frac{4}{9}$ gives $\frac{4}{9}x^2 + \frac{16}{81} = \frac{416}{81}$. Therefore $\frac{4}{9}x^2 = \frac{400}{81}$, or $x^2 = \frac{100}{9}$. Since x must be positive, we have $x = \frac{10}{3}$, so the desired quantity is 13.

2-3 The equation $xz + 2xy = x^2 + 5y(z - 3y)$ is equivalent to the equation $z = Ax + By$ for integers A and B with certain restrictions on the ordered triple solutions. One such restricted solution is the ordered triple $(C, 10, D)$. Find $A + B + C + D$.
[Answer: 134]

Separating all terms with z yields $xz - 5yz = x^2 - 2xy - 15y^2$. Factoring both sides yields $z(x - 5y) = (x - 5y)(x + 3y)$. We can divide both sides by $x - 5y$ to produce $z = x + 3y$. However, this restricts any solution where $x - 5y = 0$. If a restricted solution had $y = 10$, then $x = 50$ and $z = 50 + 3(10) = 80$, making the desired quantity $1 + 3 + 50 + 80 = 134$.

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Round 3: Solids & Volume

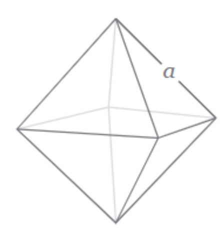
3-1 A right cylinder has a base radius of 3 cm and the furthest distance from a point on one base to a point on the opposite base is 10 cm. The volume of the cylinder is $V\pi$ cm^3. What is the value of V?
[Answer: 72]

The diameter of the base is 6, forming a leg of a right triangle with the height of the cylinder being the length of the other leg and the hypotenuse having length 10. Therefore the height of the cylinder is 8, making the volume $\pi(3^2)(8) = 72\pi$. Therefore the desired quantity is 72.

3-2 A particular hemisphere has the property that its volume is x cubic units, its radius is $\frac{1}{2}x$ units, and its surface area is y square units. What is the value of y?
[Answer: 9]

From the problem description we can set up $\frac{2}{3}\pi\left(\frac{1}{2}x\right)^3 = x$, so $\frac{1}{12}\pi x^3 = x$, and $\pi x^2 = 12$. The surface area of the hemisphere is $3\pi\left(\frac{1}{2}x\right)^2 = \frac{3}{4}\pi x^2 = \frac{3}{4}(12) = 9$, which is our desired quantity.

3-3 A regular octahedron is a three-dimensional figure composed of 8 equilateral triangles. The volume of a regular octahedron is ka^3, where a is the length of one edge of the octahedron and k is a real constant. The volume of a regular octahedron with edge length $\frac{1}{k}$ is $\frac{p}{q}$, where p and q are positive integers with no common factors greater than 1. Find $p+q$.
[Answer: 11]



Since one half of the octahedron is a regular square pyramid, the height of half the octahedron can be found using $a^2 = \left(\frac{\sqrt{2}}{2}a\right)^2 + h^2$, and therefore $h = \frac{\sqrt{2}}{2}a$. The volume of the octahedron is therefore twice the volume of one pyramid, or $2\left(\frac{1}{3}\right)(a^2)\frac{\sqrt{2}}{2}a = \frac{\sqrt{2}}{3}a^3$.

The volume of a regular octahedron with edge length $\frac{3}{\sqrt{2}}$ is $\frac{\sqrt{2}}{3}\left(\frac{3}{\sqrt{2}}\right)^3 = \frac{27\sqrt{2}}{6\sqrt{2}} = \frac{9}{2}$, making the desired quantity $9 + 2 = 11$.

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Round 4: Radical Expressions and Equations

4-1 Solve for $x > 8$: $\sqrt{x-8} = \frac{x-8}{11}$
[Answer: 129]

Since $\frac{x-8}{\sqrt{x-8}} = \sqrt{x-8}$, we have $\sqrt{x-8} = 11$, and therefore $x = 11^2 + 8 = 129$.

4-2 If $\sqrt[6]{72000000} = (\sqrt{a})(\sqrt[3]{b})$ for positive integers $a > 1$ and $b > 1$, find the sum of the maximum possible value of a and the maximum possible value of b.
[Answer: 3200]

Note that the quantity is equivalent to $10\sqrt[6]{72}$, or $10\left(3^{\frac{1}{3}}\right)\left(2^{\frac{1}{2}}\right)$. The maximum value of a would then be $2 * 10^2 = 200$, and the maximum value of b would be $3 * 10^3 = 3000$, making the desired quantity $200 + 3000 = 3200$.

4-3 What is the largest integer value of $p \leq 2024$ such that $\sqrt{x} + \sqrt{5} = \sqrt{x+p}$ has an integer solution for x?
[Answer: 2015]

Squaring both sides of the equation yields $x + 5 + 2\sqrt{5x} = x + p$. Therefore $p - 5 = 2\sqrt{5x}$, or $x = \frac{1}{5}\left(\frac{p-5}{2}\right)^2$. If x is an integer, then $p - 5$ must be even and a multiple of 5. Therefore p must be five more than a multiple of 10. The largest such value less than or equal to 2024 is 2015.

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Round 5: Polynomials and Advanced Factoring

5-1 What is the smallest integer for which the polynomial $f(x) = x^3 - 10x^2 - 1$ has a positive value?
[Answer: 11]

Note that $f(x) = x^2(x - 10) - 1$. This quantity is negative for any integer value $x \leq 10$. Therefore the smallest integer value of x that produces a positive value for $f(x)$ is 11.

5-2 If the polynomial $x^4 - 65x^2 + p$, where p is a constant, has four integer zeros, find the sum of all possible values of p.
[Answer: 848]

To have four integer zeros, the polynomial must factor into $(x + a)(x - a)(x + b)(x - b) = (x^2 - a^2)(x^2 - b^2)$. Therefore we need pairs of perfect squares that add to 65. There are two: $(1{,}64)$ and $(16{,}49)$. The possible values of p are therefore $(1)(64) = 64$ and $(16)(49) = 784$, making the desired quantity $64 + 784 = 848$.

5-3 If the polynomial $f(x) = x^3 - 18x^2 + kx - 96$, where k is a constant, has the property that its three real zeros form an arithmetic sequence, what is the value of k?
[Answer: 88]

Let the zeros of the polynomial be $a - b, a, a + b$. The sum of these zeros is $3a$ and the product is $a(a^2 - b^2)$. From the equation we see that the sum of the zeros is 18, making $a = 6$. Since the product of the zeros is 96, we know $a^2 - b^2 = 16$. Applying Vieta we know $k = a(a - b) + a(a + b) + (a - b)(a + b) = 2a^2 + a^2 - b^2 = 2(36) + 16 = 88$.

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Round 6: Counting and Probability

6-1 In how many ways can I arrange the letters of "GNOME" if the O must appear before the E (though it does not need to be next to it)?
[Answer: 60]

Unrestricted there are $5! = 120$ ways to arrange the letters of "GNOME". However only half of these arrangements will have the O appearing before the E, making the desired quantity 60.

6-2 On any given day, Mr. Forgette gives homework in his class with probability p and Mr. Bearse gives homework in his class with probability q, and their homework assigning is independent. The probability that both teachers give homework on one day is $\frac{1}{18}$, and the probability that Mr. Forgette gives homework but Mr. Bearse does not on one day is $\frac{1}{6}$. The probability that neither give homework one day is $\frac{a}{b}$, wher a and b are integers with no common factors greater than 1. Find $a + b$.
[Answer: 19]

Let $P(F)$ be the probability that Mr. Forgette gives homework and let $P(B)$ be the probability that Mr. Bearse gives homework. We have $P(F \cap B) = \frac{1}{18}$ and $P(F \cap B^C) = \frac{1}{6}$. Therefore $P(B) = P(F \cap B) + P(F \cap B^C) = \frac{2}{9}$. Since $P(F \cap B) = P(F)P(B)$, we have $\frac{1}{18} = \frac{2}{9}P(F)$, and so $P(F) = \frac{1}{4}$. Finally, $P(F^C \cap B^C) = P(F^C)P(B^C) = \left(\frac{3}{4}\right)\left(\frac{7}{9}\right) = \frac{7}{12}$, making the desired quantity $7 + 12 = 19$.

6-3 Wonka's Wilder Genes, a genetics testing company, has a test that is 80% effective in detecting the presence (or non-presence) of a certain gene. However, when someone tests positive, there is a note in the results saying that since the gene only appears in p percent of the population, the probability the person has the gene is only 50%. Find the value of p.
[Answer: 20]

Let A represent the event of having gene and let B be the probability of testing positive for the gene. The probability $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2}$. Using Bayes' Theorem or a diagram, we see this means $\frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A^C)P(A^C)}$. Note that $P(A) = \frac{p}{100}$ and $P(B|A) = .8$ and

$P(B|A^C) = .2$ Therefore we have $\frac{.8(.01p)}{.8(.01p)+.2(1-.01p)} = \frac{1}{2}$, or $\frac{8p}{200+6p} = \frac{1}{2}$, which can be solved to yield $p = 20$.

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Match 6
Team Round

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1. An arithmetic sequence of positive integer angle measures starts at 6°, and there are exactly two pairs of distinct complementary angle measures in the sequence. What is the largest angle measure in the sequence less than 180°?
[Answer: 162]

Two complementary angles in this sequence will satisfy the equation $6 + ad + 6 + bd = 90$, or $a + b = \frac{78}{d}$, where a, b, and d are all integers and $0 \leq a < b$. Setting $d = 78$ yields $a + b = 1$, which has only one solution pair, $(0,1)$. Setting $d = 39$ yields $a + b = 2$ which also has only one solution pair, $(0,2)$. However, setting $d = 26$ yields $a + b = 3$ which has two solution pairs, $(0,3)$ and $(1,2)$. Therefore d must be 26, as any smaller values of d produces too many solution pairs. Setting $6 + 26n < 180$ yields a maximum value of $n = 6$, making the desired quantity $6 + (6)(26) = 162$.

2. The graph of $2024y^2 + xy + x = 2024$ is of two lines which intersect at (a, b). Find $a + b$.
[Answer: 4047]

This can be solved by isolating y using the quadratic formula. Alternatively, setting $xy + x + 2024y^2 - 2024 = 0$ and factoring yields $(y+1)\big(x + 2024(y-1)\big) = 0$. This produces two equations for y: $y = -1$ and $y = -\frac{1}{2024}x + 1$. Setting these two values equal yields $-1 = -\frac{1}{2024}x + 1$, which when solved yields $x = 4048$. Therefore the point of intersection is $(4048, -1)$, making the desired quantity $4048 - 1 = 4047$.

3. Recall that for a convex polyhedron composed of F faces with E edges and V vertices, $F + V = E + 2$. A particular polyhedron has the property that each vertex is formed from 4 equilateral triangles and one regular pentagon. How many faces will this polyhedron have?
[Answer: 92]

Let t be the number of triangular faces and p be the number of pentagonal faces. Therefore $F = t + p$. The number of vertices is equal to the number of pentagonal vertices or $\frac{1}{4}$ the number of triangular vertices, so $V = 5p = \frac{3}{4}t$, or $t = \frac{20}{3}p$. The number of edges of the polyhedron is one half the total number of edges of all polygons, or $E = \frac{1}{2}(3t + 5p)$. Using the relationship between all these quantities and substituting $t = \frac{20}{3}p$ yields $\frac{20}{3}p + p + 5p = 10p + \frac{5}{2}p + 2$, which when solved yields $p = 12$. Consequently $t = 80$, making the total number of faces $80 + 12 = 92$.

4. If the function $f(x) = \sqrt{x} + \sqrt{x - 201}$ includes the integer pair (a, b), find the sum of all possible values of b.
[Answer: 268]

To produce an integer pair, a must be a perfect square and $a - 201$ must also be a perfect square. One way to solve this is to find a perfect square $(u + v)^2 = u^2 + 2uv + v^2$ such that $2uv + v^2 = 201$. This ensures $x = (u + v)^2$ and $x - 201 = u^2$. First note that v must be odd; otherwise $2uv + v^2$ would be even. Looking for integer pairs (u, v) where $2uv + v^2 = 201$ and therefore $2u = \frac{201}{v} - v$, we find only $(100,1)$ and $(32,3)$. Therefore $f(100) = 101 + 100 = 201$ and $f(32) = 35 + 32 = 67$, making the desired quantity $201 + 67 = 268$.

5. A cubic polynomial $f(x)$ has all integer coefficients less than 100 and a positive leading coefficient. If $f(1) = 4$, $f(2) = 10$, and $f(3) = 20$, find the largest possible sum of the absolute values of the coefficients.
[Answer: 212]

Assuming $f(x) = ax^3 + bx^2 + cx + d$, we have $a + b + c + d = 4$ (e_1), $8a + 4b + 2c + d = 10$ (e_2), and $27a + 9b + 3c + d = 20$ (e_3). This yields $e_3 - e_2 = 19a + 5b + c = 10$, and $e_2 - e_1 = 7a + 3b + c = 6$. Finally subtracting these last two equations yields $12a + 2b = 4$, or $b = 2 - 6a$. We then substitute $7a + 3(2 - 6a) + c = 6$ to get $c = 11a$, and then $a + 2 - 6a + 11a + d = 4$ yields $d = 2 - 6a$. Our coefficients in terms of a are therefore $\{a, 2 - 6a, 11a, 2 - 6a\}$. If the coefficients are integers less than 100, then $a = 9$, and the desired quantity is $9 + 52 + 99 + 52 = 212$.

6. On a standardized multiple choice exam on which each question had five options, $\frac{3}{10}$ of all takers got the hardest question correct. A team of analysts believes that there was a proportion of test takers p who knew the correct answer, and another proportion of test takers q who were "educated guessers" who were able to get the question right after narrowing the options down to two. The team is assuming that everyone else who got the question correct guessed randomly out of the five options. The values p and q satisfy the equation $Ap + Bq = 1$, where A and B are integers. Find $A + B$.
[Answer: 11]

Since all of the test takers in the proportion p got the question right, $\frac{1}{2}$ the test takers in the proportion q got the question right, and $\frac{1}{5}$ in the proportion $1 - p - q$ to the question right, we have $p + \frac{1}{2}q + \frac{1}{5}(1 - p - q) = \frac{3}{10}$. Distributing and combining yields $\frac{4}{5}p + \frac{3}{10}q + \frac{1}{5} = \frac{3}{10}$, or $8p + 3q = 1$, making the desired quantity $8 + 3 = 11$.