FAIRFIELD COUNTY MATH LEAGUE 2023-2024
Match 5
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Fractions and Exponents

1-1 The product of two-sevenths and eleven-thirtieths, subtracted from the sum of one-third and three-fifths, is $\frac{a}{b}$, where a and b are positive integers with no common factors greater than 1. Find $a + b$.

1-2 If n is a real number such $\left(\frac{3^n}{9}\right)^{n-3} = 9$, find the sum of all possible values of 3^n.

1-3 If x and n are positive numbers such that $x^n + x^{-n} = 5$, find the value of $x^{3n} + x^{-3n}$.

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Round 2: Rational Expressions and Equations

2-1 If $y = \frac{3x-7}{2x+5}$ and $z = \frac{y+10}{4y-1}$, then there exist integers a, b, c, and d that share no common factors greater than 1 such that $z = \frac{ax+b}{cx+d}$. Find the value of $a + b + c + d$.

2-2 For how many positive integer values of $n, 4 \leq n \leq 60$, is the fraction $\frac{n-3}{n+3}$ in simplest form?

2-3 Artist Lotso Monet asks his friend to watch his gallery for 8 weeks. In return, Lotso will give his friend \$1200 plus one of his paintings, which is valued at d dollars. Unfortunately, the friend has an emergency and has to leave after watching the gallery for only n weeks. Lotso compensates his friend proportionally with \$200 and the painting. If n and d are both positive integers, find the sum of the highest and lowest possible values of d.

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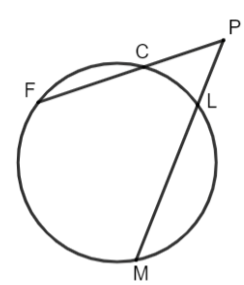
Round 3: Circles

3-1 Points F, C, M, and L lie on the circumference of a circle, and chords $\overline{FM}$ and $\overline{CL}$ intersect at point P. If $m\,\widehat{FC} = 60°$, $m\angle MPL = 40°$, and $\widehat{CM} \cong \widehat{ML}$, find the measure of $\widehat{FL}$ in degrees.

3-2 On a large clock on a tower, the minute hand is 12 inches long and the hour hand is 10 inches long. In a fixed interval of time (less than one hour), the tip of the minute hand traces out $\widehat{AB}$ and the tip of the hour hand traces out $\widehat{CD}$. If, at the end of this interval, $\widehat{CD}$ has a length of 5 inches, what is the length in inches of $\widehat{AB}$?

3-3 Refer to the diagram. Line segment $\overline{PF}$ intersects the circle at points F and C, and line segment $\overline{PM}$ intersects the circle at points M and L. $ML = 2FC$, $PL = 8$, and PC and FC are integers. If n is the number of possible values of FC and x is the largest possible value of FC, find $n + x$.



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Match 5
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Round 4: Quadratic Equations & Complex Numbers

4-1 The quadratic equation $(1+2i)z^2+7z+2-4i=0$ has two solutions z_1 and z_2, where $|z_1|<|z_2|$. For z_1, $|z_1|=\frac{a\sqrt{b}}{c}$ where $a, b,$ and c are positive integers where a and c have no common factors greater than 1 and b has no perfect square factors greater than 1. Find $a+b+c$.

4-2 A quadratic function $f(z)=z^2-5z+a+bi$, where a and b are real numbers, has zeros of $2+pi$ and $q-6i$ for some real numbers p and q. Find $a+b$.

4-3 Consider the quadratic function $g(z)=az^2+2iz-5$, where a is a nonzero integer and $|a|\leq 10$. It is known that $g(z)$ has complex zeros with rational coefficients. Find the sum of all possible values of a.

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Round 5: Trigonometric Equations

5-1 If x is an angle in Quadrant I and $\sin^2(x) + \cos^2(x) + \tan^2(x) = 10000$, find $\sec(x)$.

5-2 If $0 \leq x \leq \frac{\pi}{4}$ and $\cos(x) + \sin(x) = 1.2$, then $\cos(x) - \sin(x) = \frac{\sqrt{a}}{b}$, where a and b are positive integers and a has no perfect square factors greater than 1. Find $a + b$.

5-3 For $0 \leq x \leq 2\pi$, the sum of all possible values of x that solve the equation $2\sin(2x) = \tan(x)$ is $k\pi$ for some integer k. Find the value of k.

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Match 5
Individual Section

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Round 6: Sequences & Series

6-1 A sequence is defined recursively: $a_1 = 2, a_2 = 3$, and for all $n > 2$, $a_n = 2a_{n-2} - a_{n-1}$. How many of the first 50 terms are positive?

6-2 The third term of an arithmetic series is 9, which is three times the sum of the first three terms. What is the 30th term of the series?

6-3 An infinite geometric series has a common ratio r and infinite sum $S > 0$. The sum of the second term and r^2S exceeds the value of the first term by 40%. If S is an integer and only the first two terms of the series are integers, find the least possible value of S.

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Match 5
Team Round

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1. A proper fraction $\frac{a}{b}$ where a and b have no common factors greater than 1 has the property that multiplying the numerator by 7 and increasing the denominator by 40 will increase the value of the fraction by 40%. Find the sum of all possible values of a.

2. Given the equation $\frac{x+k}{x-4} - \frac{3x-5}{2x-1} = \frac{10x+30}{2x^2-9x+4}$, the value of k is such that $x = 4$ is an extraneous solution. Given this value of k, what is the non-extraneous solution of the equation for x?

3. A dog is on a leash tied to the corner of a shed shaped like a rectangular prism that is 20 feet long and w feet wide. The leash is 16 feet long. If the dog has as much area to roam as it would if it could roam freely in a circle at the end of a 14-foot-long leash, what is the value of w?

4. A local math league is selling protractors to raise funds for busing. Students on the team produce a model that says that when protractors are priced at \$2.50 each, they expect to sell 140 of them. Additional research indicates that every price increase of \$.50 will result in selling 10 fewer protractors. Based on this model, find the maximum number of *cents* in revenue that students will be able to earn from this fundraiser.

5. Consider right triangle FCL with right angle C. Point M lies on $\overline{CL}$ such that $\overline{FM}$ is an angle bisector. If $FL = k(FC)$ for some integer k, $CM = 5$, and $FC < 6$, find the least possible value of k.

6. An infinite geometric series with the first two terms $90 + 78 + \cdots$ has an infinite sum S. The sum of the first n terms of an arithmetic series with the first two terms $90 + 78 + \ldots$ is N. Find the minimum value of $S - N$.