FAIRFIELD COUNTY MATH LEAGUE 2023-2024
Match 5
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Fractions and Exponents

1-1 The product of two-sevenths and eleven-thirtieths, subtracted from the sum of one-third and three-fifths, is $\frac{a}{b}$, where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 64]

Setting up the computation yields $\frac{1}{3} + \frac{3}{5} - \left(\frac{2}{7}\right)\left(\frac{11}{30}\right) = \frac{14}{15} - \frac{11}{105} = \frac{87}{105} = \frac{29}{35}$, so the desired quantity is $29 + 35 = 64$.

1-2 If n is a real number such $\left(\frac{3^n}{9}\right)^{n-3} = 9$, find the sum of all possible values of 3^n.
[Answer: 84]

Rewriting the equation in terms of powers of 3 yields $(\frac{3^n}{3^2})^{n-3} = 3^2$, so $(3^{n-2})^{n-3} = 3^2$. Therefore $(n-2)(n-3) = 2$, which in standard form yields $n^2 - 5n + 4 = 0$, which factors into $(n-1)(n-4) = 0$. Therefore the solutions are $n = 1$ and $n = 4$, making the desired quantity $3^1 + 3^4 = 3 + 81 = 84$.

1-3 If x and n are positive numbers such that $x^n + x^{-n} = 5$, find the value of $x^{3n} + x^{-3n}$.
[Answer: 110]

Cubing both sides of the equation yields $x^{3n} + 3x^{2n}x^{-n} + 3x^n x^{-2n} + x^{-3n} = 125$, and simplifying the left side of the equation produces $x^{3n} + 3x^n + 3x^{-n} + x^{-3n} = 125$. Note that this means $x^{3n} + x^{-3n} + 3(5) = 125$, and therefore $x^{3n} + x^{-3n} = 110$.

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Round 2: Rational Expressions and Equations

2-1 If $y = \frac{3x-7}{2x+5}$ and $z = \frac{y+10}{4y-1}$, then there exist integers a, b, c, and d that share no common factors greater than 1 such that $z = \frac{ax+b}{cx+d}$. Find the value of $a + b + c + d$.
[Answer: 43]

Substituting the expression in terms of x for all values of y in the expression for z yields $z = \frac{\frac{3x-7}{2x+5}+10}{4\left(\frac{3x-7}{2x+5}\right)-1}$, and multiplying both the numerator and denominator by $2x + 5$ yields $z = \frac{3x-7+10(2x+5)}{4(3x-7)-(2x+5)}$, which simplifies to $z = \frac{23x+43}{10x-33}$, making the desired quantity $23 + 43 + 10 - 33 = 43$.

2-2 For how many positive integer values of $n, 4 \leq n \leq 60$, is the fraction $\frac{n-3}{n+3}$ in simplest form?
[Answer: 19]

First note that n must be even, for if n is odd, both the numerator and denominator will produce even numbers and the fraction will not be in simplest form. Additionally, if n is a multiple of 3, both the numerator and denominator will be multiples of 3 and again not in simplest form. Therefore n must be an even number that is not a multiple of three (i.e. not a multiple of 6). There are 29 positive even numbers in the indicated range, and 10 of them are multiples of 6. Therefore there are $29 - 10 = 19$ values of n that meet the criteria.

2-3 Artist Lotso Monet asks his friend to watch his gallery for 8 weeks. In return, Lotso will give his friend \$1200 plus one of his paintings, which is valued at d dollars. Unfortunately, the friend has an emergency and has to leave after watching the gallery for only n weeks. Lotso compensates his friend proportionally with \$200 and the painting. If n and d are both positive integers, find the sum of the highest and lowest possible values of d.
[Answer: 7200]

From the problem we can set up $\frac{n}{8}(1200 + d) = 200 + d$. Distributing the quantity on the left side yields $150n + \frac{dn}{8} = 200 + d$, and rewriting this equation to isolate d yields $d = \frac{150n-200}{1-\frac{n}{8}}$, which is equivalent to $d = \frac{400(3n-4)}{8-n}$. Note that d is not an integer when $n = 1$ or $n = 2$, but the value of $n = 3$ yields $d = 400$. The value of d continues to increase with n, and so when $n = 7$, we have $d = 6800$, which is the largest possible value of d. Therefore the desired quantity is

$400 + 6800 = 7200$.

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Round 3: Circles

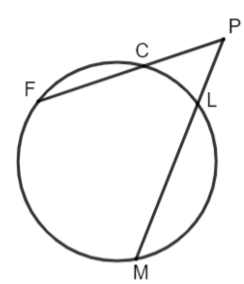
3-1 Points F, C, M, and L lie on the circumference of a circle, and chords $\overline{FM}$ and $\overline{CL}$ intersect at point P. If $m\,\widehat{FC} = 60°$, $m\angle MPL = 40°$, and $\widehat{CM} \cong \widehat{ML}$, find the measure of $\widehat{FL}$ in degrees.
[Answer: 260]

Because the vertical angles $\angle MPL$ and $\angle FPC$ subtend arcs $\widehat{FC}$ and $\widehat{ML}$, we have $40 = \frac{60+m\widehat{ML}}{2}$, so $m\,\widehat{ML} = 20°$, and therefore $m\,\widehat{CM} = 20°$, and $m\,\widehat{FL} = 360° - 60° - 20° - 20° = 260°$.

3-2 On a large clock on a tower, the minute hand is 12 inches long and the hour hand is 10 inches long. In a fixed interval of time (less than one hour), the tip of the minute hand traces out $\widehat{AB}$ and the tip of the hour hand traces out $\widehat{CD}$. If, at the end of this interval, $\widehat{CD}$ has a length of 5 inches, what is the length in inches of $\widehat{AB}$?
[Answer: 72]

In a fixed length of time less than one hour, the minute hand will trace an angle that is 12 times the measure of the angle traced out by hour hand. The length of a traced arc varies jointly with the radius of a circle and the measure of the angle traced. So if θ is the angle traced out by the hour hand, then the ratio of the length of $\widehat{AB}$ to the length of $\widehat{CD}$ is $\frac{(12)(12\theta)}{(10)\theta} = \frac{72}{5}$. Therefore, if the length of $\widehat{CD}$ is 5 inches, then the length of $\widehat{AB}$ is 72 inches.

3-3 Refer to the diagram. Line segment $\overline{PF}$ intersects the circle at points F and C, and line segment $\overline{PM}$ intersects the circle at points M and L. $ML = 2FC$, $PL = 8$, and PC and FC are integers. If n is the number of possible values of FC and x is the largest possible value of FC, find $n + x$.
[Answer: 166]



Let $PC = a$ and $FC = b$. From the problem and known theorems, we have $a(a + b) = 8(8 + 2b)$, so $a^2 + ab = 64 + 16b$. Isolating b yields $b = \frac{64-a^2}{a-16}$. The first observation is that $8 < a < 16$ so that b is positive.

Valid ordered pairs (a, b) include $(10,6)$, $(12,20)$, $(13,35)$, $(14, 66)$, and $(15, 161)$. Since there are five ordered pairs, the desired quantity is $5 + 161 = 166$.

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Round 4: Quadratic Equations & Complex Numbers

4-1 The quadratic equation $(1+2i)z^2+7z+2-4i=0$ has two solutions z_1 and z_2, where $|z_1|<|z_2|$. For z_1, $|z_1|=\frac{a\sqrt{b}}{c}$ where $a, b,$ and c are positive integers where a and c have no common factors greater than 1 and b has no perfect square factors greater than 1. Find $a+b+c$.
[Answer: 12]

Solving for values of z using the quadratic formula yields $\frac{-7\pm\sqrt{7^2-4(1+2i)(2-4i)}}{2(1+2i)}=\frac{-7\pm\sqrt{49-40}}{2+4i}=\frac{-7\pm 3}{2+4i}$. The solution with the lower absolute value is $\frac{-4}{2+4i}$, and its absolute value is $\frac{4}{\sqrt{20}}$, or in simplest radical form, $\frac{2\sqrt{5}}{5}$, making the desired quantity $2+5+5=12$.

4-2 A quadratic function $f(z)=z^2-5z+a+bi$, where a and b are real numbers, has zeros of $2+pi$ and $q-6i$ for some real numbers p and q. Find $a+b$.
[Answer: 48]

The zeros have a sum of 5, a real number, so $2+q=5$ and $p-6=0$. Therefore the zeros are $2+6i$ and $3-6i$, and $a+bi=(2+6i)(3-6i)=42+6i$, making the desired quantity $42+6=48$.

4-3 Consider the quadratic function $g(z)=az^2+2iz-5$, where a is a nonzero integer and $|a|\leq 10$. It is known that $g(z)$ has complex zeros with rational coefficients. Find the sum of all possible values of a.
[Answer: 3]

Since the zeros of g have rational coefficients, we consider the discriminant $(2i)^2-4(-5)a=-4+20a$. Since a is a nonzero integer, we look for values in the given range where this quantity produces a perfect square or the opposite of a perfect square. The set of valid ordered pairs $(a,-4+20a)$ is $(10,196)$, $(2,36)$, $(1,16)$, $(-3,-64)$, and $(-7,-144)$, making the desired quantity $10+2+1-3-7=3$.

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Round 5: Trigonometric Equations

5-1 If x is an angle in Quadrant I and $\sin^2(x) + \cos^2(x) + \tan^2(x) = 10000$, find $\sec(x)$.
[Answer: 100]

Using trigonometric identities, $1 + \tan^2(x) = \sec^2(x) = 10000$, so $\sec(x) = 100$.

5-2 If $0 \leq x \leq \frac{\pi}{4}$ and $\cos(x) + \sin(x) = 1.2$, then $\cos(x) - \sin(x) = \frac{\sqrt{a}}{b}$, where a and b are positive integers and a has no perfect square factors greater than 1. Find $a + b$.
[Answer: 19]

In the given range for x, $\cos(x) > \sin(x)$. One way to approach the problem is to square both sides of the first equation, yielding $1 + 2\sin(x)\cos(x) = 1 + \sin(2x) = 1.44$, so $\sin(2x) = .44 = \frac{11}{25}$. This means $\cos(2x) = \frac{\sqrt{504}}{25} = \frac{6\sqrt{14}}{25}$. Since $\cos(2x) = (\cos(x) + \sin(x))(\cos(x) - \sin(x))$, we have $\frac{6\sqrt{14}}{25} = \frac{6\sqrt{a}}{5b}$, so $\frac{\sqrt{a}}{b} = \frac{\sqrt{14}}{5}$, making the desired quantity $14 + 5 = 19$.

5-3 For $0 \leq x \leq 2\pi$, the sum of all possible values of x that solve the equation $2\sin(2x) = \tan(x)$ is $k\pi$ for some integer k. Find the value of k.
[Answer: 7]

We can write the equation as $4\sin(x)\cos(x) - \sin(x)\sec(x) = 0$, so $\sin(x)\,(4\cos(x) - \sec(x)) = 0$. The factor $\sin(x) = 0$ yields $x \in \{0, \pi, 2\pi\}$, and the second factor can be written as $\cos^2(x) = \frac{1}{4}$, so $\cos(x) = \pm\frac{1}{2}$, yielding $x \in \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$. The sum of the solutions is therefore 7π, making the desired quantity 7.

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Match 5
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Round 6: Sequences & Series

6-1 A sequence is defined recursively: $a_1 = 2, a_2 = 3$, and for all $n > 2$, $a_n = 2a_{n-2} - a_{n-1}$. How many of the first 50 terms are positive?
[Answer: 27]

The first 8 terms will be: $2, 3, 1, 5, -3, 13, -19, 45$ .... We can see that the first four terms are positive and then every other term will be positive as the pattern continues. This means that 4 plus half the remaining 46 terms will be positive, making the desired quantity 27.

6-2 The third term of an arithmetic series is 9, which is three times the sum of the first three terms. What is the 30th term of the series?
[Answer: 225]

If a is the first term and d is the common difference, then from the problem we have $a + 2d = 9$ and $a + a + d + a + 2d = 3a + 3d = 3$, so $a + d = 1$. Subtracting these two equations yields $d = 8$, making $a = -7$. The desired quantity is therefore $-7 + 29(8) = -7 + 232 = 225$.

6-3 An infinite geometric series has a common ratio r and infinite sum $S > 0$. The sum of the second term and r^2S exceeds the value of the first term by 40%. If S is an integer and only the first two terms of the series are integers, find the least possible value of S.
[Answer: 144]

If a is the first term, then from the problem we set up the equation $ar + r^2S = 1.4a$. Noting that $S = \frac{a}{1-r}$, we can eliminate a from the equation to make $r + \frac{r^2}{1-r} = 1.4$, which we can write as $r - r^2 + r^2 = 1.4 - 1.4r$, so $2.4r = 1.4$, so $r = \frac{7}{12}$. This means $S = \frac{a}{1-\frac{7}{12}} = \frac{12}{5}a$. If S is an integer, a must be divisible by 5. Also, if only the first two terms are integers, then the first term is divisible by 12. Therefore the least possible value of a is 60, making the least possible value of $S\ \frac{12}{5}(60) = 144$.

FAIRFIELD COUNTY MATH LEAGUE 2023-2024
Match 5
Team Round

Please write your answers on the answer sheet provided.

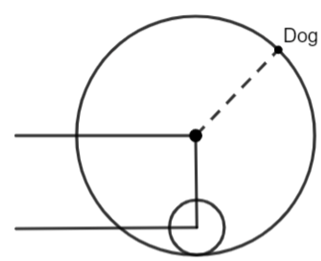
1. A proper fraction $\frac{a}{b}$ where a and b have no common factors greater than 1 has the property that multiplying the numerator by 7 and increasing the denominator by 40 will increase the value of the fraction by 40%. Find the sum of all possible values of a.
[Answer: 20]

From the problem description we can set up $\frac{7a}{b+40} = \frac{1.4a}{b}$, which can be written as $7b = 1.4b + 56$, or $5.6b = 56$. Therefore $b = 10$. This means a can be any number less than 10 that has no common factors with 10 that are greater than 1. Therefore $a \in \{1,3,7,9\}$, so the sum of all possible values of a is 20.

2. Given the equation $\frac{x+k}{x-4} - \frac{3x-5}{2x-1} = \frac{10x+30}{2x^2-9x+4}$, the value of k is such that $x = 4$ is an extraneous solution. Given this value of k, what is the non-extraneous solution of the equation for x?
[Answer: 14]

By multiplying every term by $(x-4)(2x-1)$, the equation becomes $(x+k)(2x-1) - (3x-5)(x-4) = 10x + 30$. The value $x = 4$ must be a solution to this form of the equation. Substituting $x = 4$ yields $(4+k)(7) = 70$, so $k = 6$. Now we have $(x+6)(2x-1) - (3x-5)(x-4) = 10x + 30$, which in standard form becomes $x^2 - 18x + 56 = 0$. This factors into $(x-4)(x-14) = 0$, and since $x = 4$ is extraneous, the desired solution is $x = 14$.

3. A dog is on a leash tied to the corner of a shed shaped like a rectangular prism that is 20 feet long and w feet wide. The leash is 16 feet long. If the dog has as much area to roam as it would if it could roam freely in a circle at the end of a 14-foot-long leash, what is the value of w?
[Answer: 12]

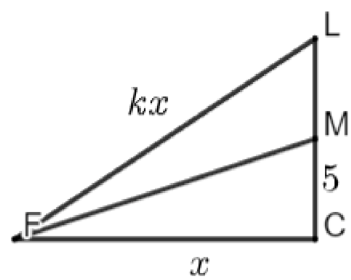


See the diagram. First we note that $w < 16$, because if $w \geq 16$, then the dog's area would just be $\frac{3}{4}(\pi)(16^2) = 192\pi < 196\pi$. Instead the dog's area is three quarters of the circular area made from the leash, plus one quarter of an additional circular area formed as if the dog's leash had radius $16 - w$. Therefore $\frac{1}{4}\pi(16-w)^2 = 4\pi$, or $(16-w)^2 = 16$, making $w = 12$.

4. A local math league is selling protractors to raise funds for busing. Students on the team produce a model that says that when protractors are priced at $2.50 each, they expect to sell 140 of them. Additional research indicates that every price increase of $.50 will result in selling 10 fewer protractors. Based on this model, find the maximum number of *cents* in revenue that students will be able to earn from this fundraiser.
[Answer: 45125]

Note first that the revenue will be (number of protractors sold)(price of each protractor). The number of protractors sold will be a linear function of the price, with a slope of $\frac{-10}{.5} = -20$ sales per dollar increase. Noting that $(2.50{,}140)$ is a point yields the linear equation $y = -20(x - 2.5) + 140$, or $y = -20x + 190$, where x is the price in dollars of the protractor and y is the number sold. Therefore the revenue equation will be $R(x) = (x)(-20x + 190) = -20x^2 + 190x$. The maximum revenue will occur when $x = \frac{-190}{2(-20)} = 4.75$, which corresponds to a revenue of $-20\left(\frac{361}{16}\right) + 190\left(\frac{19}{4}\right) = 451.25$, which in cents is 45125.

5. Consider right triangle FCL with right angle C. Point M lies on $\overline{CL}$ such that $\overline{FM}$ is an angle bisector. If $FL = k(FC)$ for some integer k, $CM = 5$, and $FC < 6$, find the least possible value of k.
[Answer: 6]



See the diagram. One approach is to use $\cos(2A)$, with A being the measure of $\angle LFC$. Therefore, since $\cos(2A) = \cos^2(A) - \sin^2(A)$ and letting $FC = x$, we have $\frac{1}{k} = \frac{x^2}{x^2+25} - \frac{25}{x^2+25}$, or $k = \frac{x^2+25}{x^2-25}$. Since $25 < x^2 < 36$, we note that when $x^2 = 36$, $k = \frac{61}{11}$, and k will continue to increase as x gets closer to 25. This gives a range of k of $k > \frac{61}{11}$, which means that the least possible integer value of k is 6.

6. An infinite geometric series with the first two terms $90 + 78 + \cdots$ has an infinite sum S. The sum of the first n terms of an arithmetic series with the first two terms $90 + 78 + \ldots$ is N. Find the minimum value of $S - N$.
[Answer: 291]

First note that $\frac{78}{90} = \frac{13}{15}$, which means $S = \frac{90}{1-\frac{13}{15}} = 675$. The partial sums of the arithmetic series will keep increasing until the terms become negative. The largest possible partial sum of the arithmetic series is $90 + 78 + 66 + 54 + 42 + 30 + 18 + 6 = 384$, making the desired quantity $675 - 384 - 291$.