FAIRFIELD COUNTY MATH LEAGUE 2023–2024
Match 4
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Basic Statistics

1-1 A set of 100 elements has a median of 6 and the property that doubling every element would not change the standard deviation of the set. What is the sum of the elements of the set?
[Answer: 600]

The only way doubling every of the set (which would double the standard deviation) would not change the standard deviation is if the standard deviation is zero. This means that all 100 elements are the same element, and would have to be 6. This makes the sum $6 * 100 = 600$.

1-2 A sequence of four numbers has a mean of 25, a median of 13, and a range of 50. What is the largest number in the sequence?
[Answer: 62]

Let the four numbers be (in decreasing order) $a, b, c,$ and d. From the problem we have $a + b + c + d = 100$, $b + c = 26$, and $a - d = 50$. From the first two equations we know $a + d = 74$, and adding this result to the last equation gives $2a = 124$, or $a = 62$.

1-3 How many sets of three distinct positive integers less than 100 have the property that their mean is equal to their median?
[Answer: 2401]

For a set of three distinct integers to have this property, the integers must form an arithmetic sequence. For example, {1,2,3} and {1,3,5} would be two sets that have this property with the smallest value 1. The set with the smallest value of 1 and the largest possible third value is {1,50,99}. This means there are 49 sets with the smallest value 1. If the smallest value is 2, the sets will range from {2,3,4} to $\{2, 50, 98\}$, making 48 sets. Similarly there are 48 sets with the smallest value of 3: {3,4,5} to {3,51,99}. This pattern continues until there is only 1 set with smallest value 96 and only 1 set with smallest value 97. Therefore the total number of sets is $49 + 2(48) + 2(47) + \cdots + 2(1) =$
$2(1 + \cdots + 49) - 49 = 49(50) - 49 = 49(50 - 1) = 49^2 = 2401$.

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Round 2: Quadratic Equations

2-1 A quadratic function $f(x)$ has one distinct real zero at $x = 6$ and a y-intercept of $(0,24)$. What is $f(15)$?
[Answer: 54]

Note that $f(x) = a(x - 6)^2$, and given $24 = a(6^2)$, we have $a = \frac{2}{3}$. Therefore $f(15) = \frac{2}{3}(15 - 6)^2 = \frac{2}{3}(81) = 54$.

2-2 The equation $2x^2 + ax + 2024 = 0$, where a is a constant, has solutions $x = p$ and $x = q$. The equation $x^2 - 68x + b = 0$, where b is a constant, has solutions $x = p + 2$ and $x = q + 2$. What is the value of b?
[Answer: 1144]

Note that the fist equation is equivalent to $x^2 + \frac{a}{2}x + 1012 = 0$, which means $pq = 1012$. From the second equation, we have $p + 2 + q + 2 = 68$, so $p + q = 64$. Therefore $b = (p + 2)(q + 2) = pq + 2(p + q) + 4 = 1012 + 2(64) + 4 = 1144$.

2-3 Consider a sequence of quadratic equations of the form $x^2 + p_n x + q_n = 0$ for $n = 1, 2, 3, \ldots$, where p_n and q_n are constants. This sequence has the property that for all $n > 1$, the zeros for the nth equation are the sum and positive difference of the zeros of the $(n - 1)$th equation. If $p_1 = -10$ and $q_1 = 9$, for how many values of n is $q_n < 10000$?
[Answer: 10]

Note that for $n = 1$ the zeros are $\{1,9\}$. This means that for $n = 2$ the zeros are $\{8,10\}$, for $n = 3$ the zeros are $\{2,18\}$ (twice the zeros of $n = 1$), and for $n = 4$ the zeros are $\{16,20\}$ (twice the zeros of $n = 2$). As this pattern continues, the values of q_n will form two geometric sequences with common ratio 4: one for the odd values of n with initial value 9 and one for even values of n with initial value 80. For the odd values of n, acceptable values of q_n are $\{9,36,144,576,2304,9216\}$ and for even values of n, acceptable values of q_n are $\{80,320,1280,5120\}$, making a total of 10 values of q_n.

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Round 3: Similarity

3-1 Right triangle ABC has legs of length 16 and 30. It is similar to triangle DEF that has one leg of length 24. What is the sum of all possible perimeters of DEF?
[Answer: 184]

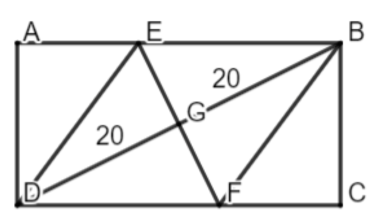
Note that the hypotenuse of ABC has length 34, and so the perimeter of ABC is 80. If the leg of length 24 corresponds to the leg of length 16, then the ratio is 1.5 and the perimeter of DEF is 120. If the leg of length 24 corresponds to the leg of length 30, then the ratio is .8 and the perimeter of DEF is 64. Therefore the desired quantity is $120 + 64 = 184$.

3-2 A sculptor is making a piece in the shape of a right square pyramid by pouring and shaping material from the bottom up. He pauses once he has a frustum with a top base area that is 36% smaller than the bottom base area. He finds he has used 305 cubic feet of material so far. How many cubic feet of material will the sculptor require for the entire sculpture?
[Answer: 625]

If the top of the frustum's area is 36% smaller than the bottom base area, then it is 64% of the bottom base's area, making the length of the sides of the top square base .8 of the length of the sides of the bottom base. This means that the pyramid is currently 20% of the final height. Therefore the current volume of the pyramid is $\left(1 - \left(\frac{4}{5}\right)^3\right) = \frac{61}{125}$ of the final volume. Setting $\frac{61}{125}x = 305$ gives a final answer of $x = 625$.

3-3 A rhombus is inscribed in a rectangle such that the rhombus and the rectangle share a diagonal of length 40. If the rhombus has area 400, what is the area of the rectangle?
[Answer: 640]

See the diagram (not drawn to scale). First note that if the area of the rhombus is 400, then $EF = 20$ and $EG = GF = 10$. Therefore $BE = DE = 10\sqrt{5}$. Note next that $\frac{BE}{EG} = \frac{BD}{DA}$, so $\frac{10\sqrt{5}}{10} = \frac{40}{DA}$, and therefore $DA = 8\sqrt{5}$. Finally $AE = \sqrt{500 - 320} = 6\sqrt{5}$. The area of the rectangle is therefore $(16\sqrt{5})(8\sqrt{5}) = 640$.



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Match 4
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Round 4: Variation

4-1 If z varies jointly as x and y and $z = 24$ when $x = 5$ and $y = 3$, find the value of xy when $z = 40$.
[Answer: 25]

Since $\frac{24}{(5)(3)} = \frac{40}{xy}$, we have that the desired quantity is $\frac{5}{8}(40) = 25$.

4-2 Studies of a new dinosaur species *Reallybigosaur fairfieldi* have shown that the length of an individual varied as the 1.1 power of its tooth length, and that its weight varies as the $\frac{10}{3}$ power of its individual length. If specimen A has a tooth length that is 8 times the tooth length of specimen B, then we would expect that specimen A would weigh k times as much as specimen B. What is the value of k?
[Answer: 2048]

Let t represent tooth length, l represent individual length, and w represent weight. Since $l \propto t^{1.1}$ and $w \propto l^{\frac{10}{3}}$, it follows that $w \propto (t^{1.1})^{\frac{10}{3}}$, or $w \propto t^{\frac{11}{3}}$. Therefore multiplying t by 8 multiplies w by $8^{\frac{11}{3}}$, or 2^{11}, or 2048.

4-3 For some real number $n > 2$, z varies directly as the nth power of x and the $(n-2)$th power of y. If $z = 720$ when $x = 6$ and $y = 2$, what is the value of z when $x = 2$ and $y = 6$?
[Answer: 80]

We have $z = kx^n y^{n-2} = \frac{k(xy)^n}{y^2}$, so $720 = \frac{k12^n}{4}$, or $k(12^n) = 2880$. Therefore $\frac{k(12^n)}{6^2} = \frac{2880}{36} = 80$.

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Round 5: Trig Expressions & DeMoivre's Theorem

5-1 If $z = 5 - i\sqrt{11}$, find the value of $|z^3|$.
[Answer: 216]

Since $|z| = \sqrt{25+11} = 6$, then $|z^3| = |z|^3 = 6^3 = 216$.

5-2 If $\sin(A) > 0$ and $\tan(A) = -2.4$, then $\sin\left(A - \frac{\pi}{3}\right) = \frac{a+b\sqrt{c}}{d}$, where a, b, and d have no common factors greater than 1 and c has no perfect square factors greater than 1. Find $a+b+c+d$.
[Answer: 46]

Note that since $\sin(A)$ is positive and $\tan(A)$ is negative, it follows that A is in quadrant 2. We also have that since $\tan(A) = -\frac{12}{5}$, $\sin(A) = \frac{12}{13}$ and $\cos(A) = -\frac{5}{13}$. Finally $\sin\left(A - \frac{\pi}{3}\right) = \sin(A)\cos\left(\frac{\pi}{3}\right) - \cos(A)\sin\left(\frac{\pi}{3}\right) = \left(\frac{12}{13}\right)\left(\frac{1}{2}\right) - \left(-\frac{5}{13}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{12+5\sqrt{3}}{26}$, making the desired quantity $12 + 5 + 3 + 26 = 46$.

5-3 Given the equation $\arctan(x+2) - \arctan(x) = \arctan\left(\frac{18}{25}\right)$, the sum of the squares of all possible values of x is $\frac{a}{b}$ where a and b are relatively prime integers. Find $a+b$.
[Answer: 77]

Since $\tan(\arctan(x+2) - \arctan(x)) = \frac{18}{25}$, we have $\frac{x+2-x}{1+(x+2)(x)} = \frac{18}{25}$, or $\frac{2}{x^2+2x+1} = \frac{18}{25}$, yielding $(x+1)^2 = \frac{25}{9}$, or $x + 1 = \pm\frac{5}{3}$. Therefore $x = -1 \pm \frac{5}{3}$, or $x \in \left\{\frac{2}{3}, -\frac{8}{3}\right\}$, and the sum of the squares of all possible values of x is $\frac{4}{9} + \frac{64}{9} = \frac{68}{9}$, making the desired quantity $68 + 9 = 77$.

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Round 6: Conic Sections

6-1 If the circle with the equation $(x+a)^2+(y+a)^2=2024$ passes through the origin, what is the value of a^2?
[Answer: 1012]

Since $(0,0)$ is a point on the circle, we have $a^2+a^2=2024$, so $a^2=1012$.

6-2 Let D represent the shortest distance from any point on the conic section $x^2+y^2-16x-30y+271=0$ to the point on the conic section $2y^2-x-4y+8=0$ that has the smallest x-value. What is D^2?
[Answer: 98]

Writing the first conic in standard form yields $(x-8)^2+(y-15)^2=18$, showing it to be a circle with center $(8,15)$ and radius $3\sqrt{2}$. Writing the second conic in standard form yields $2(y-1)^2=x-6$, showing it to be a parabola (opening in the direction of the positive x −axis) with vertex $(6,1)$. This vertex will be the point on the parabola with the smallest x-value, and the distance from the closest point on the circle to the vertex is the distance from the center of the circle to the vertex minus the radius of the circle, so $D=\sqrt{2^2+14^2}-3\sqrt{2}=10\sqrt{2}-3\sqrt{2}=7\sqrt{2}$, so $D^2=98$.

6-3 A 10-meter by 6-meter rectangle is inscribed in an ellipse. The midpoint of each six meter side is one meter away from the nearest vertex of the ellipse. The distance in meters between the midpoint of each ten meter side and the nearest vertex is $\frac{a\sqrt{b}-c}{d}$, where a, c, and d are relatively prime positive integers and b is a positive integer with no perfect square factors greater than 1. Find $a+b+c+d$.
[Answer: 73]

Assume the ellipse is oriented horizontally and therefore can be modeled by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b$. Since the distance from the center of the rectangle and a vertex of the major axis is 6 (five meters for half the length of the rectangle and one additional meter), we have $\frac{x^2}{36}+\frac{y^2}{b^2}=1$. Note also that the corners of the rectangle lie on the ellipse, so $\frac{5^2}{36}+\frac{3^2}{b^2}=1$, so $b^2=\frac{9*36}{11}$, so $b=\frac{18}{\sqrt{11}}=\frac{18\sqrt{11}}{11}$. Therefore the distance between the midpoint of one of the ten meter sides and a minor axis vertex is $\frac{18\sqrt{11}}{11}-3=\frac{18\sqrt{11}-33}{11}$, making the desired quantity $18+11+33+11=73$.

FAIRFIELD COUNTY MATH LEAGUE 2023-2024
Match 4
Team Round

Please write your answers on the answer sheet provided.

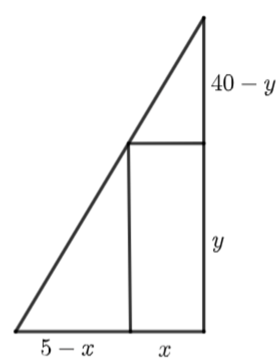
1. Set A consists of five distinct positive integers. Set B is created by tripling every element of set A and then subtracting 12 from every element. The mean of set B is the same as that of set A, but the median of set B is six less than the median of set A. The range of set B is 20% of the square of the range of set A. What is the second largest element of set B?
[Answer: 12]

If x is the mean of set A, then the mean of set B $3x - 12$. Setting $x = 3x - 12$ yields $x = 6$, which is the mean of both sets. If the median of set A is y, then the median of set B is $3y - 12$. Setting $3y - 12 = y - 6$ yields $y = 3$, so the median of set A is 3. If the range of set A is z, then the range of set B is $3z$. Setting $3z = .2z^2$ yields $z = 15$ (since $z = 0$ is extraneous), making the range of set A equal to 15. Set A therefore consists of five distinct positive integers with a median of 3, a mean of 6 (and thus a sum of 30), and a range of 15. This means set A must be $\{1,2,3,8,16\}$. The desired quantity is therefore $3(8) - 12 = 12$.

2. The quadratic equation $az^2 + bz = 8 - 15i$, where a is a nonzero complex constant and b is a real constant, has one distinct complex zero z_0. If $|z_0| = \frac{1}{3}$, find $|b|$.
[Answer: 102]

First note that $z_0 = -\frac{b}{2a}$. Therefore, $|z_0| = \frac{|b|}{2|a|} = \frac{1}{3}$, so $\frac{|b|}{|a|} = \frac{2}{3}$. Furthermore, $a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) = 8 - 15i$, so $-\frac{b^2}{4a} = 8 - 15i$. Therefore $\frac{|b|^2}{4|a|} = 17$, so $\frac{|b|}{4}\left(\frac{|b|}{|a|}\right) = 17$, and so $\frac{|b|}{4}\left(\frac{2}{3}\right) = 17$, meaning $|b| = 17 * 6 = 102$.

3. A rectangle is inscribed in a right triangle such that the triangle and the rectangle share a right angle, and one vertex of the rectangle lies on the hypotenuse of the right triangle. The triangle has legs of length 5 and 40, and the rectangle has area 32. Find the sum of the squares of all possible lengths of one diagonal of the rectangle.
[Answer: 1105]



See the diagram. If the side of the rectangle parallel to the leg of length 5 has length x and the side of the rectangle parallel to the leg of length 40 has length y, then we have $xy = 32$ and, by similarity, $\frac{40-y}{x} = \frac{y}{5-x}$. This means $(40 - y)(5 - x) = 200 - 40x - 5y + xy = xy$, or $8x + y = 40$. Substituting $y = \frac{32}{x}$, we have $8x + \frac{32}{x} = 40$, or in standard form, $8x^2 - 40x + 32 = 0$, which simplifies to $x^2 - 5x + 4 = 0$, yielding $x = 1$ or $x = 4$, This gives possible ordered pairs (x, y) of (1,32) or (4,8), making the desired quantity $1^2 + 32^2 + 4^2 + 8^2 = 1105$.

4. Assume y varies inversely as the nth power of x. If $y = 112$ when $x = 3$ and $y = 7$ when $x = 24$, then the value of y when $x = 16$ is $\frac{a\sqrt[3]{b}}{c}$, where a, b, and c are positive integers, a and c are relatively prime, and b contains no perfect cube factors greater than 1. Find $a + b + c$.
[Answer: 37]

Setting up $(112)(3^n) = (7)(24^n)$ yields $8^n = 16$, so $n = \frac{4}{3}$. Then $(7)\left(24^{\frac{4}{3}}\right) = y\left(16^{\frac{4}{3}}\right)$, so $y = 7\left(\frac{3}{2}\right)^{\frac{4}{3}} = \frac{7\left(3\sqrt[3]{3}\right)}{2\sqrt[3]{2}}$, which we can rationalize by multiplying the numerator and denominator by $\sqrt[3]{4}$ to get $\frac{21\sqrt[3]{12}}{4}$, making the desired quantity $21 + 12 + 4 = 37$.

5. How many complex numbers $z = a + bi$, where a and b are both positive, have the property that $z^{2024} = z$?
[Answer: 505]

Let $z = rcis(\theta)$. Therefore $(rcis(\theta))^{2024} = rcis(\theta + 2k\pi)$ where k is a nonnegative integer. Since z is in quadrant 1, $r = 1$ (since $r^{2024} = r$) and $0 < \theta < \frac{\pi}{2}$. Setting up $2024\theta = \theta + 2k\pi$ yields $\theta = \frac{2k\pi}{2023}$. Therefore setting $\frac{2k\pi}{2023} < \frac{\pi}{2}$ gives $k < \frac{2023}{4} = 505.75$, giving 505 possible values.

6. A hyperbola whose equation can be written in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ has one asymptote with the equation $y = 2x + 3$ and one focus at $(8,9)$. Find the value of $h + k + a^2$.
[Answer: 17]

Note from the equation that the hyperbola is oriented horizontally, meaning the focus shares its y −coordinate with the center, and so $k = 9$. Setting $9 = 2h + 3$ yields $h = 3$. This means that $c = 5$ and so $a^2 + b^2 = 25$. We also note that since the slope of the asymptote is 2, $\frac{b^2}{a^2} = 4$, making $b^2 = 4a^2$. Therefore $a^2 + 4a^2 = 25$, yielding $a^2 = 5$. This makes the desired quantity $3 + 9 + 5 = 17$,