FAIRFIELD COUNTY MATH LEAGUE 2023–2024
Match 3
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Decimals and Base Notation

1-1 For how many integers $n, 2 \leq n \leq 50$, is $\frac{1}{n}$ a terminating decimal?
[Answer: 11]

Note that there are 49 possible values of n. The decimal will terminate for any value of n that has only prime factors of 2 and 5. This includes $n \in \{2,4,5,8,10,16,20,25,32,40,50\}$, so the answer is 11.

1-2 If b is an integer base such that $301_b + 58_{10} = 413_b$, what is the value of 10_b in base 10?
[Answer: 7]

Writing the equation as a polynomial in b yields $3b^2 + 1 + 58 = 4b^2 + b + 3$, which in standard form gives $b^2 + b - 56 = 0$, which factors into $(b+8)(b-7) = 0$, making the desired value $b = 7$.

1-3 For an integer base $x > 10$, where A is the digit for ten in that base, the sequence $13_x, 13.A_x, 14.8_x$ is arithmetic. If this arithmetic sequence continues, the next integer term will have value y. Express the sum $x + y$ as a numeral in base 3.
[Answer: 1012]

Setting the differences between consecutive terms equal to each other gives $x + 4 + \frac{8}{x} - \left(x + 3 + \frac{10}{x}\right) = x + 3 + \frac{10}{x} - (x + 3)$, which simplifies to $1 - \frac{2}{x} = \frac{10}{x}$, or $1 = \frac{12}{x}$, so $x = 12$. In base twelve, the subsequent terms would be $15.6, 16.4, 17.2, 18$. This means that the next integer is 18_{12}, or 20 in base ten. Therefore $x + y = 32$ in base ten, or in base three 1012, which is the desired result.

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Round 2: Word Problems

2-1 If 3 bakers could decorate 4 cakes in 5 minutes, how many minutes would it take 10 bakers to decorate 80 cakes? Assume bakers work at a constant pace independently.
[Answer: 30]

One baker can produce $\frac{4}{3}$ cakes in 5 minutes, or $\frac{4}{15}$ cakes per minute. Therefore 10 bakers produce $\frac{40}{15} = \frac{8}{3}$ cakes per minute. Therefore it would take $\frac{80}{\frac{8}{3}} = 30$ minutes to bake 80 cakes.

2-2 Jen, Kaitlyn, and Debbie are comparing pet rock collections. They find that the ratio of pet rocks owned by Jen, Kaitlyn, and Debbie is $5: 2: 3$, respectively. Jen decides to give 85 total of her pet rocks to Kaitlyn and Debbie so they can all have the same number. How many pet rocks do they have combined?
[Answer: 510]

The first ratio in the problem has 10 total parts. This is equivalent to a ratio of $15: 6: 9$, which has 30 total parts. Jen has to give 5 of these parts to Jen and Kaitlyn, which is equivalent to 85 rocks, which means each part in this ratio represents 17 rocks. Therefore the group has a total of $(17)(30) = 510$ rocks.

2-3 An empty bus starts its route by picking up some passengers on Avon Street. It travels to Brookfield Lane, where it drops off a third of its passengers, but picks up two more than twice the number that it dropped off. From there it heads to Canton Drive, where it drops off 1 more than one-third of its passengers and picks up one-fourth of one more than it dropped off. Finally at Danbury Road it drops off all 22 of its passengers. How many passengers did the bus pick up on Avon Street?
[Answer: 21]

Let x be the number of passengers picked up on Avon Street. At Brookfield Lane, it drops off $\frac{1}{3}x$ but picks up $2\left(\frac{1}{3}x\right) + 2$ passengers, giving a total of $\frac{4}{3}x + 2$ passengers on the bus. At Canton Drive it drops off $\frac{1}{3}\left(\frac{4}{3}x + 2\right) + 1$ or $\frac{4}{9}x + \frac{5}{3}$ passengers and picks up $\frac{1}{4}\left(\frac{4}{9}x + \frac{5}{3} + 1\right)$ or $\frac{1}{9}x + \frac{2}{3}$ passengers. This means the bus ended up with $\frac{4}{3}x + 2 -$

$\left(\frac{4}{9}x + \frac{5}{3}\right) + \frac{1}{9}x + \frac{2}{3}$, or $x + 1$ passengers, which equals 22, making the original number of passengers 21.

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Round 3: Polygons

3-1 There exists a regular $k-$gon such that the number of diagonals is exactly 180 times the number of sides. What is the value of k?
[Answer: 363]

Setting $\frac{k(k-3)}{2} = 180k$, we have that $\frac{k-3}{2} = 180$, making $k = 363$.

3-2 Consider a regular $n-$gon for which the degree measure of one angle is divisible by 9. What is the largest possible value of n?
[Answer: 40]

One angle of a regular polygon is $180 - \frac{360}{n}$. If divided by 9, we get $20 - \frac{40}{n}$. The largest value of n for which this expression evaluates to an integer is $n = 40$.

3-3 An equilateral quadrilateral with perimeter 40 is inscribed in an equiangular quadrilateral with area 120 such that the diagonals of the equilateral quadrilateral are parallel with the sides of the equiangular quadrilateral. What is the square of the perimeter of the equiangular quadrilateral?
[Answer: 2560]

An equilateral quadrilateral is a rhombus, so let this rhombus have semi-diagonals a and b. We then have $a^2 + b^2 = 100$. An equiangular quadrilateral is a rectangle, so it would have sides $2a$ and $2b$, so $4ab = 120$, or $2ab = 60$. Therefore $a^2 + 2ab + b^2 = 160$, or $(a + b)^2 = 160$. The perimeter of the rectangle is $2(2a + 2b) = 4a + 4b$, and it's square is $(4a + 4b)^2 = 16(a + b)^2$, or $16(160) = 2560$.

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Round 4: Function and Inverses

Note: the inverse f^{-1} of a function is not necessarily a function.

4-1 If $g(x+1) = 3x+7$ and $g(k) = 10$, find $g(5k)$.
[Answer: 34]

Let $k = x+1$. Then $x = k-1$ and $g(k) = 3(k-1)+7 = 3k+4$. If $3k+4 = 10$, then $k = 2$, and $g(5k) = g(10) = 34$.

4-2 If $f(x)$ is a linear function with y-intercept $(0,4)$ and $f^{-1}(x)$ crosses the y −axis at $(0,-2)$, find$(f+f^{-1})(60)$.
[Answer: 152]

Note that $(-2,0)$ is another point on $f(x)$. This makes $f(x) = 2x+4$. Since $f(60) = 124$ and we can solve to find $f(28) = 60$, this means $(f+f^{-1})(60) = 124+28 = 152$.

4-3 Consider the functions $f(x) = \frac{a}{3x-b}$ and $g(x) = \sqrt{3x-c}$ with real constants a, b, and c. If $(f \circ g)^{-1}$ has a domain of $\left(-\infty, -\frac{1}{2}\right] \cup (0, \infty)$ and a range of $\left[\frac{5}{3}, 7\right) \cup (7, \infty)$, find $(f \circ g)^{-1}\left(\frac{2}{7}\right)$.
[Answer: 42]

Note that $f \circ g$ has a domain of $\left[\frac{5}{3}, 7\right) \cup (7, \infty)$ and a range of $\left(-\infty, -\frac{1}{2}\right] \cup (0, \infty)$. The function $f \circ g$ would have the structure $\frac{a}{3\sqrt{3x-c}-b}$, so the domain has restrictions $3x - c \geq 0$ and $3\sqrt{3x-c} - b \neq 0$. From the domain, we know $x \geq \frac{5}{3}$, which makes $c = 5$, and subsequently since $x \neq 7$, we can solve that $b = 12$, giving us $f \circ g = \frac{a}{3\sqrt{3x-5}-12}$. Noting that the negative part of the range occurs where $x \in \left[\frac{5}{3}, 7\right)$ and that the maximum value of the function over this interval would occur when $x = \frac{5}{3}$, we can solve for a since $\frac{a}{-12} = -\frac{1}{2}$, so $a = 6$. Therefore $f \circ g = \frac{6}{3\sqrt{3x-5}-12}$. Last setting this function equal to $\frac{2}{7}$ and solving for x gives $21 = 3\sqrt{3x-5} - 12$, or $\sqrt{3x-5} = 11$, making $x = 42$.

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Round 5: Exponents & Logarithms

5-1 If $\log_2 x = 6$, $\log_2 y = x$, and $\log_y z = \frac{3}{8}$, find $\log_2 z$.
[Answer: 24]

Note $2^6 = 64 = x$, $2^{64} = y$, and $(2^{64})^{\frac{3}{8}} = 2^{24} = z$, we have $\log_2(2^{24}) = 24$.

5-2 If m and n are positive integers less than 100 such that $\frac{36^{m-1}}{6^{3n+2}} = 6$, what is the largest possible value of n?
[Answer: 63]

Since $\frac{6^{2m-2}}{6^{3n+2}} = 6$, we have $2m - 2 - (3n + 2) = 1$, or $2m - 3n = 5$. Note that $(4,1)$ is an ordered pair for (m, n) that solves this equation, and that additional solutions will be of the form $(4 + 3k, 1 + 2k)$ for integer values of k. Setting $4 + 3k < 100$, we have $k < 32$, making $k = 31$ the largest possible value of k, and subsequently $1 + 2(31) = 63$ is the largest possible value of n.

5-3 The equation $16^x - 8^{x+\frac{2}{3}} = 6 * 4^{x+2}$ has the solution $x = a + \log_2 b$, where a and b are positive integers and b is odd. Find $a + b$.
[Answer: 5]

This problem can be written as $(4^x)^2 - 4 * 8^x = 96 * 4^x$. We can divide the equation by 4^x to get $4^x - 4 * 2^x = 96$. Letting $u = 2^x$, this can be written as a quadratic equation in standard form as $u^2 - 4u - 96 = 0$. This equation has solutions $u = 12$ and $u = -8$, but since the latter is extraneous, we have $2^x = 12$, or $x = \log_2(12) = 2 + \log_2 3$, making the desired quantity $2 + 3 = 5$.

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Round 6: Matrices

6-1 If the matrix $\begin{bmatrix} 3 & -2 \\ k & 8 \end{bmatrix}$ does not have an inverse, find the determinant of the matrix $\begin{bmatrix} k & 2 \\ k & k \end{bmatrix}$.
[Answer: 168]

Given $3(8) - (-2k) = 0$, we have $-2k = 24$, so $k = -12$. This makes the desired quantity $(-12)(-12) - 2(-12) = 144 + 24 = 168$.

6-2 If $\begin{bmatrix} 4 & 3 & 0 \\ m & 5 & 1 \\ 5 & 2 & m \end{bmatrix}$ is a singular matrix, the sum of the possible values of m is $\frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 23]

The determinant of the matrix is computed using $4(5m - 2) - 3(m^2 - 5)$, and since the matrix is singular, the determinant is 0 and consequently $3m^2 - 20m - 7 = 0$. This factors into $(m - 7)(3m + 1) = 0$, so the sum of all possible values of m is $7 - \frac{1}{3} = \frac{20}{3}$, making the desired quantity $20 + 3 = 23$.

6-3 Consider matrix $A = \begin{bmatrix} 5 & x \\ 2 & 7 \end{bmatrix}$, where x is an integer. If the matrix A^{-1} has all integer entries, find the maximum possible value of the sum of the entries of A^{-1}.
[Answer: 8]

In order for the inverse of the given matrix to have all integer entries, the given matrix must have a determinant of 1 or -1. If the determinant is 1, then $x = 17$ and the inverse is $\begin{bmatrix} 7 & -17 \\ -2 & 5 \end{bmatrix}$, which has an entry sum of -7. If the determinant is -1, then $x = 18$ and the inverse is $\begin{bmatrix} -7 & 18 \\ 2 & -5 \end{bmatrix}$, which has an entry sum of 8, which is the desired quantity.

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Match 3
Team Round

Please write your answers on the answer sheet provided.

1. Your computer programmer friend has asked you to take care of her dog while she is on a vacation. She texts you on the day she leaves: “Your pay is in the safe. Good luck.” When you arrive at the home, you find a cryptic note by a safe in the kitchen: $FED_x - BED_x = ME_y$. On the back of the note are two scrawls: “$y = 3x + 8$" and “Combination: $xxyy$”. You realize that x and y are integer bases and you know that when a base is larger than 10, letters are used to represent values larger than 9, with A = 10, B = 11, etc. If this combination is a four-digit number, with the digits of x in base ten being the first two digits and the digits of y in base ten being the second two, what is the combination to the safe?
[Answer: 1965]

Converting the equation into a polynomial in x and y gives $15x^2 + 14x + 13 - (11x^2 + 14x + 13) = 22y + 14$, or $4x^2 = 22y + 14$. Substituting $y = 3x + 8$ gives $4x^2 = 66x + 190$, or in standard form, $2x^2 - 33x - 95 = 0$. This factors into $(x - 19)(2x + 5) = 0$, meaning $x = 19$. Consequently $y = 3(19) + 8 = 65$, making the desired quantity 1965.

2. Mike and Andrew are competing in a problem solving race to see who can solve a certain number of problems the fastest. Andrew starts out solving the problems at a consistent pace that is $33\frac{1}{3}\%$ faster than Mike’s pace. However, after solving half his problems, Andrew blows a neuron and his pace slows to half of what it was originally, while Mike’s pace stays consistent. After the race has gone on for two hours, Mike overtakes Andrew and stays ahead through the last fifteen problems to win the race. On average, how many seconds did Mike spend on each problem?
[Answer: 160]

Let n be the total number of problems, x be the number of problems Andrew solved between blowing a neuron and being overtaken by Mike, and let m represent Mike’s problem solving rate. We have $\frac{\frac{n}{2}}{\frac{4}{3}m} + \frac{x}{\frac{2}{3}m} = \frac{\frac{n}{2}+x}{m}$, or $\frac{3}{8}n + \frac{3}{2}x = \frac{n}{2} + x$, or $\frac{1}{8}n = \frac{1}{2}x$, or $n = 4x$. This means that when Mike overtook Andrew, they had both solved $\frac{3}{4}$ of the total number of problems. This means there were 60 total problems, and Mike had solved 45 problems in 2 hours or 120 minutes. Mike’s average rate was therefore $\frac{8}{3}$ minutes per problem, and $\frac{8}{3}(60) = 160$ seconds.

3. What is the largest possible value of n such that the degree measure of one angle of a regular n-gon is exactly 2 degrees greater than the degree measure of one angle of a regular m-gon?
[Answer: 32220]

Set $180 - \frac{360}{n} - \left(180 - \frac{360}{m}\right) = 2$, which simplifies to $\frac{180}{m} - \frac{180}{n} = 1$, or $180n - 180m = nm$. Solving for

m produces $m = \frac{180n}{180+n} = \frac{n}{1+\frac{n}{180}}$. If $n = 180k$, then $m = 180\left(\frac{k}{k+1}\right)$. The largest value of k that will produce an integer value for m is 179. This makes the largest possible value of $n = 180(179) = 180^2 - 180 = 32400 - 180 = 32220$.

4. Consider the functions $f(x) = 5 + \frac{18}{2+e^x}$ and $g(x) = \log_2(\log_3(4x+1) - 4)$. The function $f(g(x))$ has a domain of $x > a$ and a range of $b < y < c$ for real numbers $a, b,$ and c. What is $a + b + c$?
[Answer: 39]

Note that the domain of $g(x)$ is bound by the restriction $\log_3(4x+1) - 4 > 0$, which means $x > 20$. The range of $g(x)$ is $(-\infty, \infty)$, which means the range of $e^{f(x)}$ is $(0, \infty)$, and so the range of $f(g(x))$ is $(5,14)$, making the desired quantity $20 + 5 + 14 = 39$.

5. The sum of all real solutions to the equation $(\log_9(x^2))^2 + 3\log_x(9) = 1 + 6\log_3(x)$ is $\frac{a}{b}$, where a and b are positive integers with no common factors greater than 1. Find $a + b$.
[Answer: 2200]

Note that $(\log_9(x^2))^2 = 4(\log_9 x)^2$, $\log_x(9) = \frac{\log_9(9)}{\log_9(x)} = \frac{1}{\log_9(x)}$, and $6\log_3(x) = 12\log_9(x)$, we have $4(\log_9(x))^2 + \frac{3}{\log_9(x)} - 12\log_9(x) - 1 = 0$. Letting $u = \log_9(x)$, this can be written as a polynomial in standard form, $4u^3 - 12u^2 - u - 3 = 0$. This factors into $(4u^2 - 1)(u - 3) = 0$, or $(2u-1)(2u+1)(u-3) = 0$. Therefore, $\log_9(x) \in \left\{\frac{1}{2}, -\frac{1}{2}, 3\right\}$, so $x \in \left\{3, \frac{1}{3}, 729\right\}$. Thus the sum of all solutions is $3 + \frac{1}{3} + 729 = \frac{2197}{3}$, making the desired quantity $2197 + 3 = 2200$.

6. The transpose of an $m\ x\ n$ matrix A, noted A^T, is an $n\ x\ m$ matrix in which row 1 of A is column 1 of A^T, row 2 of A is column 2 of A^T, etc. Consider matrix $A = \begin{bmatrix} 4 & x \\ 3 & y \end{bmatrix}$, where x and y are positive integers, and matrix $B = AA^T$. If $\det(A) < 0$, $\det(B) = 4$, and the entries of B have a sum of 625, find the value of x.
[Answer: 14]

Note the matrix $A^T = \begin{bmatrix} 4 & 3 \\ x & y \end{bmatrix}$, and $B = \begin{bmatrix} 4 & x \\ 3 & y \end{bmatrix}\begin{bmatrix} 4 & 3 \\ x & y \end{bmatrix} = \begin{bmatrix} 16 + x^2 & 12 + xy \\ 12 + xy & 9 + y^2 \end{bmatrix}$. Since $\det(A) < 0$ and we can see that $\det(A) = \det(A^T)$, it follows that $\det(A) = 4y - 3x = -2$. The sum of the entries of B is $49 + x^2 + 2xy + y^2$, which means $7^2 + (x+y)^2 = 25^2$. Since x and y are positive integers, this means $x + y = 24$. Solving the linear system for x and y yields $x = 14$ and $y = 10$, making $x = 14$ the desired quantity.