FAIRFIELD COUNTY MATH LEAGUE 2023–2024
Match 1
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Percentages

1-1 What is the positive difference between 123% of 123 and 77% of 77?
[Answer: 92]

This can be directly computed as $(1.23)(123) - (.77)(77)$, or we could notice that this is of the form $\frac{1}{100}((123)^2 - (77)^2) = \frac{1}{100}(123 + 77)(123 - 77) = \frac{1}{100}(200)(46) = 2(46) = 92.$-y

1-2 If n is a four digit number (with no 0 digits) such that the hundreds digit is 50% larger than the tens digit and the units (ones) digit is $33\frac{1}{3}\%$ greater than the sum of the thousands digit and the tens digit, find the difference between the largest and smallest possible values of n.
[Answer: 3004]

From the description, it is known that the tens digit is even and the sum of the tens digit and thousands digit is a multiple of 3. If the tens digit is 2, the hundreds digit is 3, the thousands digit can be 1 (the smallest possible) and the ones digit will be 4, making $n = 1324$. The largest possible thousands digit is 4, since this allows the tens digit to be 2 making the ones digit 8. The thousands digit cannot be any larger without any of the digits getting greater than 9. This largest value of n is 4328. This makes the desired quantity $4328 - 1324 = 3004$.

1-3 Positive numbers x and y have the properties that their product is 10 and the quantity of x increased by y percent is equal to the quantity of y increased by $(2x)\%$. Find the value of $100(x^2 + y^2)$.
[Answer: 2001]

Writing the description as a system of equations yields $xy = 10$ and $x(1 + \frac{y}{100}) = y(1 + \frac{2x}{100})$. Distributing on both sides of the second equation and substituting $xy = 10$ yields $x + \frac{1}{10} = y + \frac{1}{5}$, or $x - y = \frac{1}{10}$. The desired quantity can be easily obtained by squaring both sides, yielding $x^2 - 2xy + y^2 = \frac{1}{100}$, or $x^2 + y^2 = \frac{1}{100} + 20 = \frac{2001}{100}$, making the desired quantity 2001.

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Round 2: Solving Equations

2-1 Solve for x: $\frac{1}{12}x - 1 = \frac{1}{60}x + \frac{1}{15}$
[Answer: 16]

Multiplying the equation by 60 clears the fractions and makes $5x - 60 = x + 4$, or $4x = 64$, yielding $x = 16$.

2-2 If x and a are positive integers less than 100 that satisfy the equation $2(4x - 2) + a = 7(a - 1) + 3x$, what is the largest possible value of a?
[Answer: 83]

Distributing coefficients and collecting terms yields the equation $6a - 5x = 3$. Note that $a = 3$ and $x = 3$ is a solution. Additional solutions can be found by adding 5 to a and 6 to x. The largest value of x occurs at $3 + (16)(6) = 99$. This corresponds with a largest possible value of a of $3 + (16)(5) = 83$.

2-3 If the equations $a, b, c,$ and d are constants such that the equations $x + a = 0, x^2 + x + b = 0$, $x^3 + x^2 + x + c = 0$, and $3x^2 - 8x + 4x^3 = d$ all share a solution for x, then there exists integers $P, Q,$ and R such that $d = Pa + Qb + Rc$. Find $P + Q + R$.
[Answer: 8]

Multiplying the third equation by 4 yields $4x^3 + 4x^2 + 4x + 4c = 0$. Subtracting the second equation yields $4x^3 + 3x^2 + 3x - b + 4c = 0$. Finally subtracting eleven times the first equation yields $4x^3 + 3x^2 - 8x - 11a - b + 4c = 0$. This makes $d = 11a + b - 4c$, making the desired quantity $11 + 1 - 4 = 8$.

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Round 3: Triangles and Quadrilaterals

3-1 A triangle has angle measures $(2x + 32)°, (6x - 12)°$ and $(8x)°$. What is the measure of the smallest angle of the triangle in degrees?
[Answer: 48]

Adding the three expressions together and setting equal to 180 yields $16x + 20 = 180$, making $x = 10$. This makes the angle measures $52°, 48°$, and $80°$ respectively, so the desired quantity is 48.

3-2 Triangle ABC has sides of length $7, 11$, and x. Triangle DEF is isosceles and has sides of length $7, 15$, and x. Triangle GHI is equilateral and has the same perimeter as triangle ABC. The altitude of triangle GHI is $\frac{a\sqrt{b}}{c}$, where b has no perfect square factors greater than 1 and a and c have no common factors greater than 1. Find $a + b + c$.
[Answer: 16]

Starting with triangle DEF, we note that x must be either 7 or 15 since the triangle is isosceles. However, x cannot be 7 as the sides would not satisfy the triangle inequality, so $x = 15$. This means ABC has a perimeter of 33, and so GHI has sides of length 11. This makes the length of the altitude $\frac{11\sqrt{3}}{2}$, and the desired value is $11 + 3 + 2 = 16$.

3-3 A right trapezoid has a height of 20 units, an area of 360 square units, and a smaller diagonal of length 25 units. What is the length in units of the longer diagonal?
[Answer: 29]

Since the smaller diagonal forms a right triangle with the height and smaller base of the trapezoid, we can find the length of the smaller base using the Pythagorean theorem: $b_1^2 + 20^2 = 25^2$, so $b_1 = 15$. Using the area of the trapezoid then yields $360 = \frac{1}{2}(20)(15 + b_2)$, so $b_2 = 21$. The longer diagonal's length can be found similarly to that of the smaller base using the Pythagorean theorem: $20^2 + 21^2 = x^2$, which yields a value of $x = 29$.

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Round 4: Systems of Equations

4-1 Bluebird Orchards sells donuts by the box as well as pies. Each pie costs \$12 and each box of donuts costs \$8. On a particular weekend, Bluebird Orchards sold 50% more boxes of donuts than pies and for a total of \$528. How many boxes of donuts were sold that weekend?
[Answer: 33]

Let $x =$ the number of pies and $y =$number of boxes of donuts. We can set this up as a system: $12x + 8y = 528$ and $y = 1.5x$. Using substitution yields $12x + 8(1.5x) = 528$, or $24x = 528$, so $x = 22$, making the desired value $y = 33$.

4-2 The sum of all positive values of x that are part of solutions of the system $\begin{cases} y = x^2 - 10y + 6 \\ \frac{x-5}{y+1} = \frac{y-1}{x+5} \end{cases}$ can be written as $a + b\sqrt{c}$, where a, b, and c are all integers and c has no perfect square factors larger than 1. Find $a + b + c$.
[Answer: 24]

The second equation can be rewritten using cross-multiplication as $x^2 - 25 = y^2 - 1$, or $x^2 = y^2 + 24$. Using substitution yields the equation $y = y^2 + 24 - 10y + 6$, or in standard form, $y^2 - 11y + 30 = 0$. This is factorable into $(y - 5)(y - 6) = 0$, giving possible y-values of 5 and 6. If $y = 5$, then $x^2 = 49$, so $x = 7$ (positive values only), and if $y = 6$, then $x^2 = 60$, so $x = 2\sqrt{15}$. This makes the sum of all possible positive x-values $7 + 2\sqrt{15}$, making the desired quantity $7 + 2 + 15 = 24$.

4-3 Given the system $\begin{cases} x^2 + y^3 = 42 \\ y^3 + z^4 = 104 \\ x^2 + z^4 = 98 \end{cases}$, the solution is $(a\sqrt{b}, c\sqrt[3]{d}, f\sqrt[4]{g})$ where a, b, c, d, f, and g are positive integers greater than one and all quantities are in simplest radical form. Find the value of $a + b + c + d + f + g$.
[Answer: 17]

One way of handling this is to add all three equations which yields $2x^2 + 2y^3 + 2z^4 = 244$, or $x^2 + y^3 + z^4 = 122$. Each of the three equations can then be subtracted from this one to give $z^4 = 80, x^2 = 18$, and $y^3 = 24$. This makes the solution as an ordered triple $(3\sqrt{2}, 2\sqrt[3]{3}, 2\sqrt[4]{5})$, making the desired quantity $3 + 2 + 2 + 3 + 2 + 5 = 17$.

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Round 5: Right Triangles

5-1 What is the measure in degrees of an acute angle whose cotangent is three times its tangent?
[Answer: 30]

Setting $\frac{1}{tan(x)} = 3tan(x)$, we have $tan^2(x) = \frac{1}{3}$, or $tan(x) = \frac{1}{\sqrt{3}}$ (since the angle is acute), which corresponds to an angle measure of 30 degrees.

5-2 A right triangle has the property that the sine of one of its acute angles is three times the cosine of the same angle. If the triangle has an area less than 375, what is the largest possible integer length of the hypotenuse of the triangle?
[Answer: 49]

This situation can be modeled with a right triangle with legs with lengths k and $3k$. The Pythagorean theorem gives a hypotenuse length of $k\sqrt{10}$. Given the restriction on the area of the triangle, we have $\frac{1}{2}(k)(3k) < 375$, or $k^2 < 250$. This means $k < 5\sqrt{10}$, and therefore $k\sqrt{10} < 50$, making the largest possible integer length of the hypotenuse 49.

5-3 A right triangle whose side lengths form an arithmetic sequence has an area of 222 square units. What is the square of the length of the hypotenuse?
[Answer: 925]

An astute observer may notice that if a right triangle's side lengths form an arithmetic sequence, the triangle will be a multiple of a $3 - 4 - 5$ triangle. Otherwise a student can set up $s^2 + (s + d)^2 = (s + 2d)^2$ to get $s^2 - 2sd - 3d^2 = 0$, which yields $s = 3d$, which confirms the observation. Therefore we have $\frac{1}{2}(3k)(4k) = 222$, or $k^2 = 37$. The square of the hypotenuse will be $(5k)^2$ or $25k^2$, making the desired quantity $25(37) = 925$.

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Round 6: Coordinate Geometry

6-1 The point P with coordinates $(8,3)$ is rotated 90 degrees counterclockwise around the origin to make the point P'. Find the square of the distance from P to P'.
[Answer: 146]

Rotating point P counterclockwise produces new coordinates $(-3,8)$, making the square of the distance between the two points $(8-(-3))^2+(3-8)^2=121+25=146$.

6-2 Given the points $A(2,3)$, $B(6,4)$, and $C(8,9)$, there are exactly three other distinct possible locations for point D such that the four ordered pairs form the vertices of a parallelogram. Find the sum of all possible y −coordinates of point D.
[Answer: 16]

We do not want point D to be collinear with $\overline{AB}$ or $\overline{BC}$. We have two possible locations such that $\overline{AD}||\overline{BC}$: $(4,8)$ and $(0,-2)$, and two possible location such that $\overline{CD}||\overline{AB}$: $(4,8)$ and $(12,10)$, with one of those locations, $(4,8)$, fulfilling both criteria. The desired quantity is then $8+(-2)+10$, or 16.

6-3 The graph of the equation $2\sqrt{x}-\sqrt{y}=4$ shares an ordered pair with a line with intercepts of $(0,296)$ and $(74,0)$. Find the square root of the product of the coordinates of this ordered pair.
[Answer: 70]

Squaring the first equation yields $4x-4\sqrt{xy}+y=16$. The equation of the line can be written in intercept form as $\frac{x}{74}+\frac{y}{296}=1$, which is equivalent to $4x+y=296$. We can use substitution in the first equation to yield $296-4\sqrt{xy}=16$ or $\sqrt{xy}=70$, which is our desired quantity.

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Team Round

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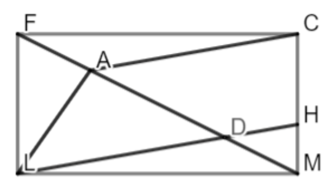
T-1 In an arithmetic sequence, the third term is 40% greater than the first term. Consequently, the 12^{th} term is p percent greater than the first term. Find the value of p.
[Answer: 220]

If the first term is p and the third term is $1.4p$, then the common difference must be $.2p$. Therefore, the 12^{th} term of the sequence is $p + 11(.2p) = p + 2.2p$, making the desired quantity 220.

T-2 Find the positive integer value of k such that the equation $\frac{x}{x-3} + \frac{5}{x+8} = \frac{x^2+20x+k}{x^2+5x-24}$ has no solution for x.
[Answer: 41]

First note that x cannot be either 3 or -8. Multiplying the equation by $x^2 + 5x - 24$ gives $x^2 + 13x - 15 = x^2 + 20x + k$, or $-7x = k + 15$. Considering values of k that occur when x is a value that is excluded as a possible solution gives us $k = -36$ when $x = 3$ and $k = 41$ when $x = -8$, so the desired quantity is the only positive value of k, 41.

T-3 Consider rectangle $FCML$, with point A on diagonal $\overline{FM}$ such tht $FA = \frac{1}{4}FM$, and point H on $\overline{CM}$ such that $\overline{AC}||\overline{LH}$. If the area of trapezoid $ACHL$ is 84 square units, find the area of rectangle $FCML$ in square units.
[Answer: 144]



See the diagram. Let point D be the intersection of $\overline{LH}$ and $\overline{FM}$. Points A and D are ¼ of the distance between the opposite sides of the rectangle. Since the area of triangle FAC is $\frac{1}{2}(FC)(\frac{1}{4}FL)$ and the area of triangle FAL is $\frac{1}{2}(FL)(\frac{1}{4}LM)$, the sum of the areas of the two triangle is $\frac{1}{4}$ the area of $FCML$. We can use similarity to determine that since D is $\frac{1}{4}$ the distance from $\overline{LM}$ to $\overline{FC}$ and $\frac{3}{4}$ the distance from $\overline{FL}$ to $\overline{CM}$, then $HM = \frac{1}{3}CM$, making the area of triangle LMH $\frac{1}{2}(LM)(\frac{1}{3}CM)$, or $\frac{1}{6}$ the area of $FCML$. Therefore the total area outside the trapezoid is $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$ the area of $FCML$ and the trapezoid is therefore $\frac{7}{12}$ the area of $FCML$. The means the desired quantity is $\frac{12}{7}(84) = 144$ square units.

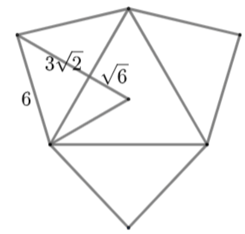
T-4 If the ordered pair (a, b) solves the system $\begin{cases} 3x + 2y = 288 \\ \frac{1}{12x} + \frac{1}{8y} = 2 \end{cases}$, find the value of ab.

[Answer: 6]

If the second equation is rewritten with the left side combined into one rational term with the least common denominator, it becomes $\frac{2y+3x}{24xy} = 2$. Substituting in the first equation yields $\frac{288}{24xy} = 2$, which means that $xy = 6$, which is our desired quantity.

T-5 An equilateral hexagon has a perimeter of 36 and has angles that alternate between measuring 90 degrees and 150 degrees. The furthest distance from the center of the hexagon to a point on the perimeter is $\sqrt{a} + b\sqrt{c}$, where $a, b,$ and c are integers and a and c have no perfect square factors greater than 1. Find $a + b + c$.
[Answer: 11]

See the diagram. This hexagon is constructed with three isosceles right triangles whose hypotenuses form an equilateral triangle. Tracing the line segement from the center of the equilateral triangle to one of the further three vertices makes an isosceles right triangle with hypotenuse 6 and a 30-60-90 triangle with a longer leg of length $3\sqrt{2}$, which means the shorter leg will have a length of $\sqrt{6}$. Therefore the total distance will be $\sqrt{6} + 3\sqrt{2}$, making the desired quantity $6 + 3 + 2 = 11$.



T-6 A circle lies in Quadrant I and is mutually tangent to the lines $y = \frac{3}{4}x$ and $y = \frac{4}{3}x$, and the x-coordinate of the point of tangency to $y = \frac{3}{4}x$ is 12. The circle has a radius of $\frac{a}{b}$, where a and b are integers with no common factors greater than 1. Find $a + b$.
[Answer: 22]

See the diagram. Because the two lines are symmetric across $y = x$, the center of the circle (point D) lies on $y = x$, so the coordinates of the center are the same. Looking first at similar right triangles ABC and DEC, let $AB = BD = k$ and $DE = r$, the radius of the circle. Therefore $BC = \frac{3}{4}BD = \frac{3}{4}k$, so $CD = \frac{1}{4}k$. Furthermore $AC = \frac{5}{4}k$. Using the fact that $\frac{AC}{AB} = \frac{DC}{DE}$, we have $\frac{\frac{5}{4}k}{k} = \frac{\frac{1}{4}k}{r}$, so $k = 5r$. Then using right triangle AED, we know $AE = \sqrt{9^2 + 12^2} = 15$ and $AD = k\sqrt{2}$, so $15^2 + r^2 = 2k^2$, or $225 + r^2 = 50r^2$, yielding $r^2 = \frac{225}{49}$, so $r = \frac{15}{7}$, making the desired quantity $15 + 7 = 22$.