FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 6
Individual Section

Please write your answers on the answer sheet provided.

<u>Round 1: Lines and Angles</u>

1-1 Angle A has the property that its measure is exactly nine times the measure of its complementary angle. Find the measure of angle A in degrees.
[Answer: 81]

Let x be the measure of Angle A in degrees. Then from the problem, $x = 9(90 - x)$, so $x = 810 - 9x$, and therefore $x = 81$.

1-2 Mr. Zucca is preparing a geometry exam. He has parallelogram $FCML$ and he wrote in that $m\angle F = (x^2 - 7x + 78)°$. He had written that one of the other angles had a measure of $(x^2 - 4x + 39)°$, but he does not remember which angle that was. He does remember that all the angle measures were supposed to be integers. Find the sum of all possible values of $m\angle F$.
[Answer: 252]

The two angles mentioned can be either congruent (if they are opposite) or supplementary (if they are adjacent). Setting the measures equal yields $x^2 - 7x + 78 = x^2 - 4x + 39$, which gives $x = 13$, making the measure of Angle F 156°. If they are supplementary, we have $2x^2 - 11x + 117 = 180$, or $2x^2 - 11x - 63 = 0$ which factors into $(x - 9)(2x + 7) = 0$ and therefore yields answers of $x = 9$ or $x = -\frac{7}{2}$. Because the angle measures are integers, the only valid answer here is $x = 9$, making the measure of angle F 96°, making the desired value $156 + 96 = 252$.

1-3 Angles $A, B, C,$ and D all have positive integer degree measures less than 180. The measure of angle B is twice that of the measure of angle A. Angle C is supplementary to angle A and angle D is supplementary to angle B. If $m\angle A < m\angle D < m\angle B < m\angle C$, find the sum of the least and greatest possible values of $m\angle A$.
[Answer: 105]

Let the measure of angle A be x. Then $m\angle B = 2x, m\angle D = 180 - 2x$, and $m\angle C = 180 - x$. Setting $m\angle A < m\angle D$ gives $x < 180 - 2x$, so $x < 60$. Setting $m\angle D < m\angle B$ gives $180 - 2x < 2x$, so $x > 45$. Setting $m\angle B < m\angle C$ gives $2x < 180 - x$, which redundantly provides $x < 60$. Therefore $45 < x < 60$, making the desired value $46 + 59 = 105$.

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Round 2: Literal Equations

2-1 In the equation $x(z-3)+5z=y(4-2z)$, $z=\frac{Ax+By}{x+Cy+D}$ for integers A, B, C, and D. Find $A-B-2C+3D$.
[Answer: 10]

Distributing all products and collecting terms with z yields $xz+2yz+5z=3x+4y$, so $z=\frac{3x+4y}{x+2y+5}$, making the desired quantity $3-4-2(2)+3(5)=10$.

2-2 The equation $x=\sqrt{y}+\sqrt{y-3}$, when solved for y, yields $y=\frac{x^A+Bx^C+D}{Ex^2}$, where A, B, C, D, and E are integers. Find $A+B+C+D+E$.
[Answer: 25]

Squaring both sides of the equation $x-\sqrt{y}=\sqrt{y-3}$ gives $x^2-2x\sqrt{y}+y=y-3$. Simplifying and isolating the remaining radical gives $\sqrt{y}=\frac{x^2+3}{2x}$, and squaring this equation yields $y=\frac{x^4+6x^2+9}{4x^2}$, making the desired quantity $4+6+2+9+4=25$.

2-3 The equation $16y=12x^5-(6+2y)x^3+96x^2-48$ is equivalent to $y=Ax^2+B$ for integers A and B, but only when x is not equal to a particular integer C. Find $A+BC$.
[Answer: 12]

Distributing all products and collecting y terms will yield $y(2x^3+16)=12x^5-6x^3+96x^2-48$. We can note that all terms divide by 2, so $y(x^3+8)=6x^5-3x^3+48x^2-24$. The right side factors by grouping, giving $y(x^3+8)=(3x^3+24)(2x^2-1)$, and so dividing both sides by x^3+8 (as long as $x\neq-2$) gives $y=6x^2-3$, making the desired quantity $6+(-3)(-2)=12$.

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Round 3: Solids & Volume

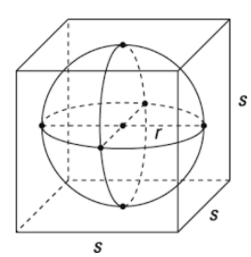
3-1 The volume of a sphere with a surface area of π units is $\frac{a}{b}\pi$ units, where a and b are positive integers with no common factors greater than 1. Find $b - a$.
[Answer: 5]

Since the volume formula of a sphere is $4\pi r^2$, setting this equal to π gives $r = \frac{1}{2}$, and then using the volume formula of $\frac{4}{3}\pi r^3$ gives a volume of $\frac{1}{6}\pi$, making the desired quantity $6 - 1 = 5$.

3-2 A triangle on the xy-plane with vertices $(0,0), (15,8)$, and $(0,44)$ is rotated around the y-axis to form a three-dimensional figure. The surface area of this figure is $k\pi$ for some integer k. Find the value of k.
[Answer: 840]

The result of the rotation will be a 3-dimensional figure in the shape of two cones meeting at a congruent circular base with a radius of 15 whose diameter lies on $y = 8$ The smaller cone has a height of 8, making the slant height $\sqrt{8^2 + 15^2} = 17$. The larger cone has a height of 36, making the slant height $\sqrt{36^2 + 15^2} = 39$. The surface area of the figure is the sum of the lateral surface area of the two cones, or $(15)(17)\pi + (15)(39)\pi$, or 840π. Therefore, the desired quantity is 840.

3-3 A sphere is inscribed in a cube (see diagram) and the shortest distance from one vertex of the cube to the surface of the sphere is 5 centimeters. The surface area of the cube in square centimeters is $a + b\sqrt{c}$, where a, b, and c are positive integers and c has no perfect square factors greater than 1. Find $a + bc$.
[Answer: 1500]



Because the radius r of the sphere is also half the length of an edge of the cube, it follows that the distance from a vertex of the cube to the midpoint of a face is $r\sqrt{2}$. We can then determine that the distance from the vertex of the cube to the center of the sphere is

$\sqrt{r^2 + 2r^2} = r\sqrt{3}$. This means that $r\sqrt{3} - r = 5$, or $r = \frac{5}{\sqrt{3}-1}$, which when rationalized becomes $r = \frac{5}{2}(\sqrt{3} + 1)$. This gives an edge of the cube a length of $5\sqrt{3} + 5$, and therefore the surface area of the cube is $6(5\sqrt{3} + 5)^2 = 600 + 300\sqrt{3}$, making the desired quantity $600 + 300(3) = 1500$.

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<u>Round 4: Radical Expressions and Equations</u>

4-1 What is the extraneous solution to the equation $2 - x = \sqrt{5x - 4}$?
[Answer: 8]

Squaring both sides of the equation yields $x^2 - 4x + 4 = 5x - 4$, which in standard form makes $x^2 - 9x + 8 = 0$. Solving this by factoring or the quadratic formula yields $x = 1$, which solves the original equation, and $x = 8$, which does not. Therefore the desired quantity is 8.

4-2 The quantity $(\sqrt[3]{4})(\sqrt{6})$ can be written in simplest radical form as $a\sqrt[b]{c}$, where a, b, and c are integers. Find the value of $2a - b + c$.
[Answer: 52]

Let $4^{\frac{1}{3}} * 6^{\frac{1}{2}} = y$, and then raising both sides to the sixth power yields $4^2 6^3 = y^6$. The left side can be rewritten as $2^7 * 3^3$, so therefore $y = 2\sqrt[6]{2 * 3^3} = 2\sqrt[6]{54}$, making the desired quantity $2(2) - 6 + 54 = 52$.

4-3 The equation $(3x + 1)^{\frac{2}{3}} + 28 = (3993x + 1331)^{\frac{1}{3}}$ has two real solutions, x_1 and x_2, where
$x_1 > x_2$. Find the value of $2x_1 + x_2$.
[Answer: 249]

Let $u = \sqrt[3]{3x + 1}$ and note that $1331 = 11^3$. The equation can be rewritten as $u^2 - 11u + 28 = 0$. This can be solved with factoring to yield $u = 7$ and $u = 4$. Substituting back $\sqrt[3]{3x + 1} = 7$ gives $x = 114$ and $\sqrt[3]{3x + 1} = 4$ gives $x = 21$, making the desired quantity $2(114) + 21 = 249$.

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Round 5: Polynomials and Advanced Factoring

5-1 The polynomials $f(x) = x^3 + k$ where k is a constant and $g(x) = 3x + 12$ share an ordered pair when $x = 2$. What is the value of k?
[Answer: 10]

Setting $f(2) = g(2)$ gives $2^3 + k = 3(2) + 12$, which when solved yields $k = 10$.

5-2 The polynomial $Ax^3 + 11x^2 + Bx - 4$ can be factored into $(3x + C)(2x^2 + 3x + D)$, where $A, B, C,$ and D are integers. what is the value of $A + 2C - B$?
[Answer: 17]

Distributing the factored polynomial yields $6x^3 + (9 + 2C)x^2 + (3D + 3C)x + CD$, and setting the terms equal to the original polynomial gives $A = 6, C = \frac{11-9}{2} = 1, D = \frac{-4}{1} = -4$, and $B = 3(-4) + 3(1) = -9$, making the desired value $6 + 2(1) - (-9) = 17$.

5-3 A polynomial function $f(x) = x^4 - x^3 + ax^2 + bx - 30$ where a and b are integers has one zero of $x = -2 + i$. Find the value of $|a| + |b|$.
[Answer: 70]

From the original polynomial we know the zeros have a sum of 1 and a product of -30. Because all the coefficients are integers, we know another zero must be $-2 - i$. These two zeros of a sum of -4 and a product of 5. This means the remaining two zeros must have a sum of 5 and a product of -6. From here we can write $f(x)$ in factored form as $f(x) = (x^2 + 4x + 5)(x^2 - 5x - 6)$, and expanding it makes $f(x) = x^4 - x^3 - 21x^2 - 49x - 30$, making our desired quantity $21 + 49 = 70$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 6x
Individual Section

Please write your answers on the answer sheet provided.

Round 6: Counting and Probability

6-1 How many different four-digit integers can I make by rearranging the numbers in 2023, including 2023? (Note: zero can not be the leading digit in a four-digit number.)
[Answer: 9]

Aside from simply listing the possibilities, we note that there are 3 possible numbers for the first digit, 3 for the second, 2 for the third, and 1 for the fourth, making $3 * 3 * 2 * 1 = 18$ possible numbers. However, since two of the digits are the same (2), this double counts the possibilities since the 2's can be switched and would still yield the same result, so the correct answer is 9.

6-2 An unfair ten-sided die has sides labled 1-10. The probability of rolling at most 4 in a single roll is $\frac{4}{9}$. The probability of rolling at least 4 in a single roll is $\frac{3}{5}$. The probability of rolling a 4 is $\frac{a}{b}$, where a and b are positive integers with no common factors greater than 1. Find $b - a$.
[Answer: 43]

Let x be the result of one roll of the die. Then $P(x \leq 4) = P(x < 4) + P(x = 4) = \frac{4}{9}$, and $P(x \geq 4) = P(x > 4) + P(4) = \frac{3}{5}$. We also know that $P(1 \leq x \leq 10) = 1 = P(x < 4) + P(x > 4) + P(x = 4)$. Adding the given probabilities together yields $P(x < 4) + P(x > 4) + 2P(x = 4) = \frac{4}{9} + \frac{3}{5} = \frac{47}{45}$, so $1 + P(x = 4) = \frac{47}{45}$, so $P(x = 4) = \frac{2}{45}$, making the desired quantity $45 - 2 = 43$.

6-3 Jess Masters wants to play a game. She places a chess piece is in the bottom left square of a 4x4 board. The piece may move one square at a time up, down, left, or right. She wants to move the piece from the bottom left square to the top right moving *only* up or right, and then she wants to move the piece back to the bottom left moving *only* down or left. In addition, in the path from the top right back to the bottom left, she does not want the piece to enter any other square in the top-most row or left-most column. How many different paths from the bottom left to the top right and back again can Jess take?
[Answer: 120]

To go from the bottom left to the top right, the piece must make six moves, three of them being up and three being right. This gives $\binom{6}{3} = 20$ total ways for the first part. Going back down, the only piece must move down first and left last, and so the possible paths including moving down twice and left twice, making $\binom{4}{2} = 6$ total ways for the second part. This makes as a total number of $20 * 6 = 120$ possible paths.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 6
Team Round

Please write your answers on the answer sheet provided.

1. Let n be the number of regions into which 10 distinct lines will divide a plane. Find the difference between the greatest and least possible values of n.
   [Answer: 45]

   Inspection shows that the fewest regions will occur when the lines are parallel, which is always one more than the number of lines; in this case, that will be 11 regions. For maximum regions, we can see that 1 line yields 2, 2 lines yield 4, 3 lines yield 7, and 4 lines yield 11. This is a quadratic progression leading to the next ordered pairs: $(5,16), (6,22), (7,29), (8,37), (9,46), (10,56)$. Therefore the desired quantity is $56 - 11 = 45$.

2. For positive real numbers x and y such that $x > y$, the equation $\frac{x}{y} - \frac{y}{x} = 4\sqrt{6}$ is true when $y = (a - b\sqrt{c})x$ where $a, b,$ and c are positive integers and c has no perfect square factors greater than 1. Find $a + \frac{c}{b}$.
   [Answer: 8]

   Letting $x = 1$, we can rewrite the equation as $\frac{1}{y} - y = 4\sqrt{6}$, which can then be written as a quadratic in standard form as $y^2 + 4\sqrt{6}y - 1 = 0$, which when solved for y with the quadratic formula gives $y = \frac{-4\sqrt{6}\pm 10}{2}$, but since y must be positive, this makes $y = 5 - 2\sqrt{6}$, making the desired value $5 + \frac{6}{2} = 8$.

3. A right square pyramid has a base with side lengths s and a height h perpendicular to the base. This pyramid has a surface area A_1. The pyramid is then sheared in half by a plane through the vertex, with the length of the plane parallel to a side of the base. One half of the pyramid is then discarded, and the surface area of the remaining half is A_2. If $s = h$, the quantity $\frac{A_1}{A_2}$ is $a - b\sqrt{c}$, where a, b, and c are positive integers and c has no perfect square factors greater than 1. Find $a + 2b + 3c$.
   [Answer: 25]

   The surface area of a square pyramid is $s^2 + 4\left(\frac{1}{2}\right)(s)\sqrt{\frac{s^2}{4} + h^2}$, which when $h = s$ simplifies to $s^2(1 + \sqrt{5})$. By shearing away half of the pyramid symmetrically, this quantity is reduced in half, but then a triangular surface of area $\frac{1}{2}sh$ (or $\frac{1}{2}s^2$ since $h = s$) is added, making a quantity $\frac{1}{2}s^2(1 + \sqrt{5}) + \frac{1}{2}s^2$. The ratio of these quantities cancels out the s^2 term, leaving $\frac{1+\sqrt{5}}{1+\frac{1}{2}\sqrt{5}}$, which after multiplying both the numerator and denominator by 2 and rationalizing becomes $6 - 2\sqrt{5}$, making the desired quantity $6 + 2(2) + 3(5) = 25$.

4. For a particular value of x, the expression $\sqrt{2x + \sqrt{2x+1}}$ is greater than the expression $\sqrt{2x+1}$ by $\frac{1}{3}$. This value of x can be written as $\frac{a}{b}$ where a and b are integers with no common factors greater than 1. Find $a + 2b$.
[Answer: 127]

Setting up the equation $\sqrt{2x + \sqrt{2x+1}} = \sqrt{2x+1} + \frac{1}{3}$, we can then square both sides to produce $2x + \sqrt{2x+1} = 2x + \frac{10}{9} + \frac{2}{3}\sqrt{2x+1}$, which simplifies to $\sqrt{2x+1} = \frac{10}{3}$. After squaring and solving for x, we have $x = \frac{91}{18}$, making the desired quantity $91 + 2(18) = 127$.

5. The Sophie Germaine Identity states $x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$. What is the largest value of the integer $k < 10000$ such that $x^4 + ky^4 = (x^2 + axy + by^2)(x^2 - axy + by^2)$ where a and b are positive integers?
[Answer: 9604]

Expanding the factored form yields $x^4 + (2b - a^2)x^2y^2 + b^2y^2$. Therefore $k = b^2 = \frac{a^4}{4}$. This means k must be the quotient of the fourth power of an even number and 4. If $a = 2m$, then $k = \frac{(2m)^4}{4} = 4m^4 < 10000$, so $m^4 < 2500$. The largest integer value of m that satisfies this inequality is $m = 7$, and $4 * 7^4 = 4(2401) = 9604$, which is the desired quantity.

6. Ivana Vinmunny has \$1 to play "Prime & Punishment". The game costs \$1 to pay. Ivana must roll a fair 20-sided die with sides labeled with the integers 1 through 20. Ivana will lose \$1 if she rolls a prime number, but will win \$1 otherwise. Ivana will keep playing forever unless she runs out of money. The probability Ivana will go broke is $\frac{a}{b}$, where a and b are integers with no common factors greater than 1. Find $10a + b$.
[Answer: 23]

First note that there are 8 prime numbers on the die: $\{2,3,5,7,11,13,17,19\}$. Therefore, the probability she will lose on a die roll is $\frac{2}{5}$. Let x be the probability that she will ultimately be down \$1 from her current amount. This will occur with certainty if she rolls a prime. If she rolls a composite, this will then occur with probability x^2 because the event will then have to occur twice. Then $x = \frac{2}{5} + \frac{3}{5}x^2$, which in standard form makes $3x^2 - 5x + 2 = 0$, which has solutions of $x = 1$ (which is extraneous since it is possible for her to not go broke) or $x = \frac{2}{3}$, which is the correct value. Therefore, the desired quantity is $10(2) + 3 = 23$.