**FAIRFIELD COUNTY MATH LEAGUE 2022–2023**
**Match 5**
Individual Section

**Please write your answers on the answer sheet provided.**

Round 1: Fractions and Exponents

1-1 How many positive integers $n$, $2 \leq n \leq 20$, have the property that there are $n-1$ different simplified proper fractions with a denominator of $n$?

1-2 The expression $\frac{2^{\frac{4}{3}}}{\left(16^{\frac{5}{6}}\right)\left(8^{-\frac{3}{5}}\right)}$ can be written as $\frac{\sqrt[a]{b}}{c}$, where $a$, $b$, and $c$ are positive integers and $b$ has no factors greater than 1 that can be written as an integer to the power of $a$. Find the value of $b^{\frac{a}{c}}$.

1-3 If $\frac{2^{12x^2+y^2}(16^x)^{x-y}}{(8^y)^{4x-y}} = 2$ for some constants $x$ and $y$, then the sum of all possible values of $\frac{81^x}{9^y}$ is $\frac{a}{b}$, where $a$ and $b$ are positive integers with no common factors greater than 1. Find the value of $a-b$.

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Round 2: Rational Expressions and Equations

2-1 The rational equation $\frac{x}{x+1}+\frac{x}{x+4}=\frac{12}{x^2+5x+4}$ has a valid rational solution $m$, but the algebra also produces an extraneous solution $n$. Find the value of $6m-2n$.

2-2 The rational expression $\frac{1}{3+\frac{1}{x+\frac{1}{2}}}$, where $x$ is a positive integer, is equivalent to a ratio of relatively prime integers where the denominator is exactly 60 more than the numerator. What is the value of $x$?

2-3 Shriya is mixing together a fruit juice drink. She starts with 600 milliliters of orange juice and she completely mixes in $x$ milliliters of pineapple juice. She drinks 200 milliliters of the mixture but then adds $2x$ milliliters of grapefruit juice. The proportion of the drink by volume now composed of pineapple juice in terms of $x$ is $\frac{x^2+Ax+B}{Cx^2+Dx+E}$. Find the value of $C(D+2A)-BE$.

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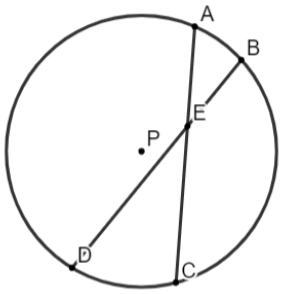
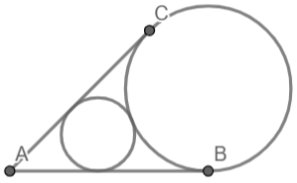
Round 3: Circles

3-1 A circle has the property that its area in square units is exactly 8 times its circumference in units. What is the length in units of the longest chord in the circle?

3-2 See the diagram, not necessarily drawn to scale. A circle with center $P$ has a radius of length 9 units and two chords $\overline{AC}$ and $\overline{BD}$ which meet at point $E$. If $m\widehat{CD} = 2m\widehat{AB}$ and $m\angle AED = 140°$, then the length of $\widehat{CD}$ is $\frac{a}{b}\pi$ units where $a$ and $b$ are positive integers with no common factors greater than 1. Find the value of $2a + b$.




3-3 See the diagram. Two circles are tangent to each other and are also tangent to line segments $\overline{AB}$ and $\overline{AC}$. If the smaller circle has an area of $9\pi$ and the larger circle has an area of $144\pi$, find $AB$.



**FAIRFIELD COUNTY MATH LEAGUE 2022–2023**
**Match 5**
Individual Section

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Round 4: Quadratic Equations & Complex Numbers

4-1 A quadratic $f(x)$ with a leading coefficient of 1 and all rational coefficients has a zero at $x = 1 - 3i$. What is the value of $f(10)$?

4-2 Let $f$ and $g$ be quadratic polynomials. $f(z)$ has all rational coefficients and a zero of $z = 3 + 4i$. $g(z)$ is of the form $g(z) = z^2 - 2iz + p + qi$ where $p$ and $q$ are real numbers and has a zero in common with $f(z)$ that is not $3 + 4i$. $|p + qi|$ can be written as $a\sqrt{b}$ where $a$ and $b$ are positive integers and $b$ has no perfect square factors greater than 1. Find $3a - b$.

4-3 A quadratic function $h$ has the form $h(z) = az^2 - 5iz + c$, where $a$ and $c$ are complex coefficients. If $a$ and $c$ are conjugates and $h\left(\frac{9i}{a}\right) = 0$, find the value of $|a|$.

**FAIRFIELD COUNTY MATH LEAGUE 2022–2023**
**Match 5**
Individual Section

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Round 5: Trigonometric Equations

5-1 If $6\cos(x) + 8 = 12$, find $36\sin^2(x) + 12$.

5-2 Consider the equation $\sec(x) - 2 = 2\tan(x) - \csc(x)$ for $x \in [0,2\pi)$. If $A$ is the largest value of $x$ that satisfies the equation and $B$ is the smallest value of $x$ that satisfies the equation, find the value of $\frac{360}{\pi}(A - B)$.

5-3 The equation $A\cos^3(x) + B\cos^2(x) + C\cos(x) + D = 0$, where $A, B, C$, and $D$ are integers with no common factors greater than 1 and $A > 0$, has the solution set $x \in \left\{\frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{5\pi}{3}, \frac{7\pi}{4}\right\}$. Find the value of $A + B + C + D$.

**FAIRFIELD COUNTY MATH LEAGUE 2022–2023**
**Match 5**
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Round 6: Sequences & Series

6-1 An arithmetic sequence has the first three terms $3, 7, 11, \ldots$. What is the average (arithmetic mean) of the first $100$ terms?

6-2 There are two infinite geometric series with the same first term $a_1 = 48$ and common ratios $r_1$ and $r_2$. For each series, the infinite sum is 12 more than five times the second term. Find the value of $\frac{1}{1-r_1-r_2}$.

6-3 There is an arithmetic series with the first term $k$ such that the sum of the first $N$ terms for all $N \geq 1$ is $kN^2$. Find the value of the 100th term of the series if $k = 10$.

**FAIRFIELD COUNTY MATH LEAGUE 2022–2023**
**Match 5**
Team Round

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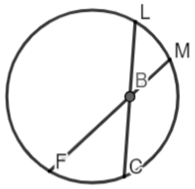
1. Let $n, a,$ and $b$ be positive integers such that $\frac{n}{2023} = \frac{1}{a+\frac{1}{b}}$. If $n \leq 10$ and $b > 1$, find the smallest possible value of $a$.

2. There are two values of the constant $a$ such that the equation $\frac{5}{ax-4} = \frac{2}{x+3}$ would have no solutions for $x$. The quadratic equation $Ma^2 + Na + P = 0$, for relatively prime integers $M, N,$ and $P$, has solutions equal to these two values of $a$. Find the value of $|M| + |N| + |P|$.

3. See the diagram (not drawn to scale), which shows a circle with two chords $\overline{FM}$ and $\overline{LC}$ that intersect at point $B$. $\overline{FL}$ is a diameter of the circle, $FB = 5$, $MB = 2$, and $m\widehat{CM} = 60°$. The area of the circle is $\frac{c}{d}\pi$ where $c$ and $d$ are positive integers with no common factors greater than 1. Find the value of $10c + d$.




4. Consider the polynomial $f(z) = z^2 + (2 - 4i)z - 3 - 10i$. If $z_0 = a + bi$, where $a$ and $b$ are integers, has the property that $f(z_0)$ lies on the real axis, what is the value of $|f(z_0)|$?

5. There are three angles $\theta, 0 \leq \theta < \frac{\pi}{2}$, such that $\sin(5\theta) = \cos(\theta)$. The sum of these angle measures in radians is $\frac{a}{b}\pi$ where $a$ and $b$ are positive integers with no common factors greater than 1. Find $2b - a$.

6. Consider a sequence where $a_0 = 5$, $a_1 = 6$, $a_2 = 7$, and for $n > 2, a_n = 2a_{n-1} - a_{n-3}$. Find the smallest value $n$ such that $a_n - a_{n-1} > 1000$.