FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 5
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Fractions and Exponents

1-1 How many positive integers n, $2 \leq n \leq 20$, have the property that there are $n-1$ different simplified proper fractions with a denominator of n?
[Answer: 8]

In order for there to be $n-1$ different simplified fractions, no proper fractions with a denominator n must be able to be simplified, so n must be prime. There are 8 primes between 2 and 20: {2,3,5,7,11,13,17,19}, making the desired value 8.

1-2 The expression $\frac{2^{\frac{4}{3}}}{\left(16^{\frac{5}{6}}\right)\left(8^{-\frac{3}{5}}\right)}$ can be written as $\frac{\sqrt[a]{b}}{c}$, where a, b, and c are positive integers and b has no factors greater than 1 that can be written as an integer to the power of a. Find the value of $b^{\frac{a}{c}}$.
[Answer: 1024]

Writing all exponential quantities as having base 2 makes the expression $\frac{2^{\frac{4}{3}}}{\left(2^{\frac{10}{3}}\right)\left(2^{-\frac{9}{5}}\right)}$, making $2^{\frac{4}{3}-\frac{10}{3}+\frac{9}{5}}$ or $2^{-\frac{1}{5}}$ or $\frac{1}{2^{\frac{1}{5}}}$, which we can change by multiplying the numerator and denominator by $2^{\frac{4}{5}}$ to get $\frac{\sqrt[5]{16}}{2}$. This makes the desired quantity $16^{\frac{5}{2}} = 2^{10} = 1024$.

1-3 If $\frac{2^{12x^2+y^2}(16^x)^{x-y}}{(8^y)^{4x-y}} = 2$ for some constants x and y, then the sum of all possible values of $\frac{81^x}{9^y}$ is $\frac{a}{b}$, where a and b are positive integers with no common factors greater than 1. Find the value of $a-b$.
[Answer: 7]

Writing all exponential quantities on the left side as having base 2 make s the expression $\frac{\left(2^{12x^2+y^2}\right)\left(2^{4x^2-4xy}\right)}{2^{12xy-3y^2}}$, which simplifies to the equation $2^{16x^2-16xy+4y^2} = 2$, so $16x^2 - 16xy + 4y^2 = 1$, which also means$(2x-y)^2 = \frac{1}{4}$, so $2x - y = \pm\frac{1}{2}$. The expression $\frac{81^x}{9^y}$ can be written as 9^{2x-y}, so this quantity may equal 3 or $\frac{1}{3}$, the sum of which is $\frac{10}{3}$, making the desired quantity $10 - 3 = 7$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 5
Individual Section

Please write your answers on the answer sheet provided.

Round 2: Rational Expressions and Equations

2-1 The rational equation $\frac{x}{x+1}+\frac{x}{x+4}=\frac{12}{x^2+5x+4}$ has a valid rational solution m, but the algebra also produces an extraneous solution n. Find the value of $6m-2n$.
[Answer: 17]

Multiplying every term by $(x+1)(x+4)$ gives $x(x+4)+x(x+1)=12$, which in standard form makes the quadratic $2x^2+5x-12=0$, which factors to make $(2x-3)(x+4)=0$. This gives solutions of $x=\frac{3}{2}$ and $x=-4$. However, $x=-4$ is extraneous. Therefore the desired quantity is $6\left(\frac{3}{2}\right)-2(-4)=17$.

2-2 The rational expression $\frac{1}{3+\frac{1}{x+\frac{1}{2}}}$, where x is a positive integer, is equivalent to a ratio of relatively prime integers where the denominator is exactly 60 more than the numerator. What is the value of x?
[Answer: 14]

Simplifying the complex fraction gives $\frac{1}{3+\frac{1}{\frac{2x+1}{2}}}=\frac{1}{3+\frac{2}{2x+1}}=\frac{1}{\frac{6x+5}{2x+1}}=\frac{2x+1}{6x+5}$. Therefore we know $6x+5=2x+1+60$, which we can solve to get $x=14$.

2-3 Shriya is mixing together a fruit juice drink. She starts with 600 milliliters of orange juice and she completely mixes in x milliliters of pineapple juice. She drinks 200 milliliters of the mixture but then adds $2x$ milliliters of grapefruit juice. The proportion of the drink by volume now composed of pineapple juice in terms of x is $\frac{x^2+Ax+B}{Cx^2+Dx+E}$. Find the value of $C(D+2A)-BE$.
[Answer: 9000]

After the initial mixing, the proportion by volume of pineapple juice in the mixture is $\frac{x}{x+600}$. The amount of pineapple juice she then consumes when she drinks 200 milliliters of the mixture is $\frac{200x}{x+600}$. After adding the additional grapefruit juice, the proportion by volume of pineapple juice in the mixture is $\frac{x-\frac{200x}{x+600}}{x+600-200+2x}$. Multiplying the numerator and denominator by $x+600$ yields $\frac{x(x+600)-200x}{(3x+400)(x+600)}$, which expands to $\frac{x^2+400x}{3x^2+2200x+240000}$. This makes the desired quantity $3(2200+2(400))-(0)(240000)=9000$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 5
Individual Section

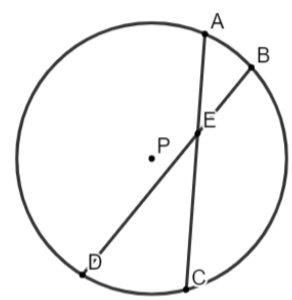
Please write your answers on the answer sheet provided.

Round 3: Circles

3-1 A circle has the property that its area in square units is exactly 8 times its circumference in units. What is the length in units of the longest chord in the circle?
[Answer: 32]

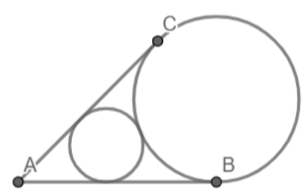
If r is the radius of the circle, then the equation described is $\pi r^2 = 8(2\pi r)$, making $r = 16$. Since the longest chord in the circle is the diameter, the desired quantity is 32.

3-2 See the diagram, not necessarily drawn to scale. A circle with center P has a radius of length 9 units and two chords $\overline{AC}$ and $\overline{BD}$ which meet at point E. If $m\widehat{CD} = 2m\widehat{AB}$ and $m\angle AED = 140°$, then the length of $\widehat{CD}$ is $\frac{a}{b}\pi$ units where a and b are positive integers with no common factors greater than 1. Find the value of $2a + b$.
[Answer: 19]

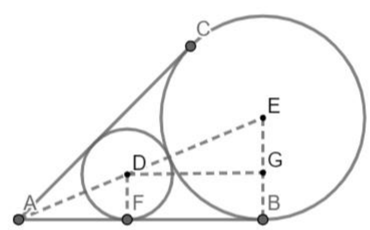


Because $m\angle AED = 140°$, we have $m\angle AEB = 40°$, which means $40 = \frac{m\widehat{AB}+m\widehat{CD}}{2} = \frac{3m\widehat{AB}}{2}$, so $m\widehat{AB} = \left(\frac{80}{3}\right)°$ and $m\widehat{CD} = \left(\frac{160}{3}\right)°$. Therefore the arclength is $\frac{160}{3} * \frac{18\pi}{360} = \frac{8}{3}\pi$, making the desired quantity $2(8) + 3 = 19$.

3-3 See the diagram. Two circles are tangent to each other and are also tangent to line segments $\overline{AB}$ and $\overline{AC}$. If the smaller circle has an area of 9π and the larger circle has an area of 144π, find AB.
[Answer: 16]



Refer to the diagram, where centers D and E have been drawn in, as well as point of tangency F and point G such that $\overline{DG}$ is parallel to $\overline{AB}$. Since $DE = 3 + 12 = 15$ and $EG = 12 - 3 = 9$, it follows that $DG^2 = 15^2 - 9^2$, so $DG = 12$. We can then use similar triangles knowing $\frac{AF}{3} = \frac{AF+12}{12}$ to find that $AF = 4$, making $AB = 16$.



FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 5
Individual Section

Please write your answers on the answer sheet provided.

Round 4: Quadratic Equations & Complex Numbers

4-1 A quadratic $f(x)$ with a leading coefficient of 1 and all rational coefficients has a zero at $x = 1 - 3i$. What is the value of $f(10)$?
[Answer: 90]

Because all of the coefficients are rational, the other zero must be $1 + 3i$, The sum of the zeros is 2 and the product is $1 - 9i^2 = 10$, so $f(x) = x^2 - 2x + 10$, and $f(10) = 100 - 20 + 10 = 90$.

4-2 Let f and g be quadratic polynomials. $f(z)$ has all rational coefficients and a zero of $z = 3 + 4i$. $g(z)$ is of the form $g(z) = z^2 - 2iz + p + qi$ where p and q are real numbers and has a zero in common with $f(z)$ that is not $3 + 4i$. $|p + qi|$ can be written as $a\sqrt{b}$ where a and b are positive integers and b has no perfect square factors greater than 1. Find $3a - b$.
[Answer: 40]

With the given information, $g(z)$ must have a zero of $3 - 4i$. Since the sum of the zeros of $g(z)$ is $2i$, it follows that the other zero of $g(z)$ must be $-3 + 6i$. The product of the zeros is $(3 - 4i)(-3 + 6i) = -9 + 18i + 12i - 24i^2 = 15 + 30i$. The magnitude of this number is $\sqrt{15^2 + 30^2} = 15\sqrt{5}$, making the desired quantity $3(15) - 5 = 40$.

4-3 A quadratic function h has the form $h(z) = az^2 - 5iz + c$, where a and c are complex coefficients. If a and c are conjugates and $h\left(\frac{9i}{a}\right) = 0$, find the value of $|a|$.
[Answer: 6]

Note that the zeros of h are $\frac{5i \pm \sqrt{-25-4ac}}{2a}$, and one of these must equal $\frac{9i}{a}$, meaning $5i + \sqrt{-25 - 4ac} = 18i$, and so $\sqrt{-25 - 4ac} = 13i$. Because a and c are conjugates, $ac = |a|^2$, so $-25 - 4|a|^2 = -169$, so $|a|^2 = 36$, and therefore $|a| = 6$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 5
Individual Section

Please write your answers on the answer sheet provided.

Round 5: Trigonometric Equations

5-1 If $6\cos(x) + 8 = 12$, find $36\sin^2(x) + 12$.
[Answer: 32]

We have $\cos(x) = \frac{2}{3}$, and so $\sin^2(x) = 1 - \frac{4}{9} = \frac{5}{9}$, making the desired quantity $36\left(\frac{5}{9}\right) + 12 = 32$.

5-2 Consider the equation $\sec(x) - 2 = 2\tan(x) - \csc(x)$ for $x \in [0,2\pi)$. If A is the largest value of x that satisfies the equation and B is the smallest value of x that satisfies the equation, find the value of $\frac{360}{\pi}(A - B)$.
[Answer: 570]

Multiplying the entire equation by $\sin(x)\cos(x)$ makes the equation $\sin(x) - 2\sin(x)\cos(x) = 2\sin^2(x) - \cos(x)$. Moving all terms to one side makes $2\sin^2(x) - \sin(x) + 2\sin(x)\cos(x) - \cos(x) = 0$, which factors into $(2\sin(x) - 1)(\sin(x) + \cos(x)) = 0$. Solutions come from $\sin(x) = \frac{1}{2}$ and $\sin(x) = -\cos(x)$, making a solution set of $x \in \left\{\frac{\pi}{6}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{4}\right\}$. This makes the desired quantity $\frac{360}{\pi}\left(\frac{7\pi}{4} - \frac{\pi}{6}\right) = \frac{360(19\pi)}{12\pi} = 570$.

5-3 The equation $A\cos^3(x) + B\cos^2(x) + C\cos(x) + D = 0$, where $A, B, C,$ and D are integers with no common factors greater than 1 and $A > 0$, has the solution set $x \in \left\{\frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{5\pi}{3}, \frac{7\pi}{4}\right\}$. Find the value of $A + B + C + D$.
[Answer: 1]

One way to approach this problem is to construct linear equations in cosine that will produce the desired solution set. The equation $\cos(x) = \frac{1}{2}$ has solutions $x \in \left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$. The equation $\cos(x) = \frac{1}{\sqrt{2}}$ has solutions $x \in \left\{\frac{\pi}{4}, \frac{7\pi}{4}\right\}$. The equation $\cos(x) = -\frac{1}{\sqrt{2}}$ has solutions $x \in \left\{\frac{3\pi}{4}, \frac{5\pi}{4}\right\}$. This makes an equation with three linear factors in cosine: $(2\cos(x) - 1)\left(\sqrt{2}\cos(x) - 1\right)\left(\sqrt{2}\cos(x) + 1\right) = 0$, when the factors are multiplied together, the equation becomes $4\cos^3(x) - 2\cos^2(x) - 2\cos(x) + 1 = 0$, making the desired quantity $4 - 2 - 2 + 1 = 1$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 5
Individual Section

Please write your answers on the answer sheet provided.

Round 6: Sequences & Series

6-1 An arithmetic sequence has the first three terms $3, 7, 11, ...$. What is the average (arithmetic mean) of the first 100 terms?
[Answer: 201]

Because the series is arithmetic, the average of the first 100 terms is also the average of the first and last term. The last term is $3 + 99(4) = 399$, so the desired quantity is $\frac{3+399}{2} = 201$.

6-2 There are two infinite geometric series with the same first term $a_1 = 48$ and common ratios r_1 and r_2. For each series, the infinite sum is 12 more than five times the second term. Find the value of $\frac{1}{1-r_1-r_2}$.
[Answer: 20]

If r is an known ratio, then we can construct the eqaution $\frac{48}{1-r} = 12 + 5(48r)$, which simplifies to $240r^2 - 228r + 36 = 0$. Note that this means the sum of the two solutions is $\frac{228}{240} = \frac{19}{20}$, which makes the desired value $\frac{1}{1-\frac{19}{20}} = 20$.

6-3 There is an arithmetic series with the first term k such that the sum of the first N terms for all $N \geq 1$ is kN^2. Find the value of the 100th term of the series if $k = 10$.
[Answer: 1990]

Note that the first term must be equal to $10(1)^2 = 10$. The sum of the first two terms must be $10(2)^2 = 40$, which means the second term must be 30. Therefore the series has a common difference of 20, and the desired value is $10 + 99(20) = 1990$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 5
Team Round

Please write your answers on the answer sheet provided.

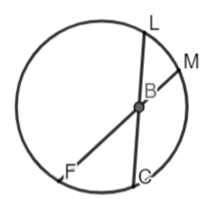
1. Let $n, a,$ and b be positive integers such that $\frac{n}{2023} = \frac{1}{a+\frac{1}{b}}$. If $n \leq 10$ and $b > 1$, find the smallest possible value of a.
[Answer: 337]

Noting that $abn + n = 2023b$ and therefore $b = \frac{n}{2023-an}$, we then notice that smaller values of a will correspond with larger values of n and that $2023 - an$ must be less than n if $b > 1$. For $n = 10$, $2023 - 10a$ must equal $1, 2,$ or 5, which means $10a$ must equal $2022, 2021,$ or 2018, none of which are possible if a is an integer. We continue decreasing values of n until we notice for $n = 6$, it is possible that $2023 - 6a = 1$, because then $6a = 2022$ and therefore $a = 337$.

2. There are two values of the constant a such that the equation $\frac{5}{ax-4} = \frac{2}{x+3}$ would have no solutions for x. The quadratic equation $Ma^2 + Na + P = 0$, for relatively prime integers $M, N,$ and P, has solutions equal to these two values of a. Find the value of $|M| + |N| + |P|$.
[Answer: 33]

First note that $x \neq -3$. Cross-multiplication produces $5x + 15 = 2ax - 8$. This will have no solutions for x if $5 = 2a$, so $a \neq \frac{5}{2}$. Additionally, since $x \neq -3$, it follows that $-15 + 15 \neq -6a - 8$, so $a \neq -\frac{4}{3}$. This leads to the quadratic equation $(2a - 5)(3a + 4) = 0$, which expands to make $6a^2 - 7a - 20 = 0$, making the desired quantity $6 + 7 + 20 = 33$.

3. See the diagram (not drawn to scale), which shows a circle with two chords $\overline{FM}$ and $\overline{LC}$ that intersect at point B. $\overline{FL}$ is a diameter of the circle, $FB = 5$, $MB = 2$, and $m\widehat{CM} = 60°$. The area of the circle is $\frac{c}{d}\pi$ where c and d are positive integers with no common factors greater than 1. Find the value of $10c + d$.
[Answer: 614]



Because $m\widehat{CM} = 60°$ and $\widehat{FL}$ is a semicircle, it follows that $m\widehat{FC} + m\widehat{ML} = 120°$ and therefore $m\angle FBC = 60°$. Also, because $\widehat{FL}$ is a semicircle, $\angle FCL$ is a right angle and therefore $m\angle FBC = 30°$. This means $BC = \frac{5}{2}$ and $FC = \frac{5\sqrt{3}}{2}$. We can find BL since $\frac{5}{2}BL = (5)(2)$, and so $BL = 4$ and $CL = \frac{13}{2}$. We can find the diameter FL using $(FL)^2 = \left(\frac{13}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2$, yielding $(FL)^2 = 61$, so $FL = \sqrt{61}$. The area of the circle is therefore $\left(\frac{\sqrt{61}}{2}\right)^2 \pi = \frac{61}{4}\pi$, making the desired quantity $10(61) + 4 = 614$.

4. Consider the polynomial $f(z) = z^2 + (2 - 4i)z - 3 - 10i$. If $z_0 = a + bi$, where a and b are integers, has the property that $f(z_0)$ lies on the real axis, what is the value of $|f(z_0)|$?
[Answer: 8]

One way to solve this is to put $f(z)$ into vertex form, making $f(z) = (z + 1 - 2i))^2 - 3 - 10i - (1 - 2i)^2 = (z + 1 - 2i)^2 - 6i$. Letting $z_0 = a + bi$ gives $f(z_0) = (a + 1 + (b - 2)i)^2 - 6i$. This quantity will have a real component of $(a + 1)^2 - (b - 2)^2$ and an imaginary component of $(2(a + 1)(b - 2) - 6)i$, which must equal zero. Therefore $(a + 1)(b - 2) = 3$. Since a and b are integers, then $a = 0$ and $b = 5$ or $a = 2$ and $b = 3$. In either case, $|(a + 1)^2 - (b - 2)^2| = |1 - 9| = 8$.

5. There are three angles $\theta, 0 \leq \theta < \frac{\pi}{2}$, such that $\sin(5\theta) = \cos(\theta)$. The sum of these angle measures in radians is $\frac{a}{b}\pi$ where a and b are positive integers with no common factors greater than 1. Find $2b - a$.
[Answer: 11]

Note that $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$. The value of $\frac{\pi}{2} - \theta$ will always be in quadrant I, but there are three cases we must consider for 5θ: 1) it is less than $\frac{\pi}{2}$, 2) it lies between $\frac{\pi}{2}$ and π, and 3) it lies between 2π and $\frac{5}{2}\pi$. For case 1 we have $5\theta = \frac{\pi}{2} - \theta$, so $\theta = \frac{\pi}{12}$. For case 2 we have $\pi - 5\theta = \frac{\pi}{2} - \theta$, so $\theta = \frac{\pi}{8}$. Finally for case 3 we have $5\theta - 2\pi = \frac{\pi}{2} - \theta$, so $\theta = \frac{5\pi}{12}$. The sum of these values is $\frac{5\pi}{8}$, making the desired quantity $2(8) - 5 = 11$.

6. Consider a sequence where $a_0 = 5$, $a_1 = 6$, $a_2 = 7$, and for $n > 2, a_n = 2a_{n-1} - a_{n-3}$. Find the smallest value n such that $a_n - a_{n-1} > 1000$.
[Answer: 17]

If we construct the first few terms of the sequence, we get $5, 6, 7, 9, 12, 17, 25\ldots$ Notice that the value of $a_n - a_{n-1}$ is the n^{th} Fibonacci number, where $F_1 = F_2 = 1$. Constructing the Fibonacci sequence until we arrive at a term greater than 1000 yields $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597\ldots$, and the last term is the 17th term of the sequence, making the desired value 17.