FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 4
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Basic Statistics

1-1 The mean absolute deviation of a set of numbers is the average (arithmetic mean) of the positive differences between each number and the arithmetic mean of the data set. If a is the mean absolute deviation of the first ten positive integers and b is the median of the first 50 positive odd integers, find the product ab.
[Answer: 125]

The mean of the first ten positive integers is $\frac{55}{10} = 5.5$, making the mean absolute deviation $\frac{2(.5+1.5+2.5+3.5+4.5)}{10} = \frac{25}{10} = 2.5$. The first 50 positive odd integers have middle values of 49 and 51, making the median 50. The desired value is then $(2.5)(50) = 125$.

1-2 A set of four numbers, three of which of are $2, 8$, and 12, has the property that its average (arithmetic mean) is equal to its median. Find the sum of all possible values of the fourth number.
[Answer: 22]

The fourth number may be the maximum value (making the median 10), the minimum value (making the median 5), or neither, making the median the average of the fourth number and 8. In the first case, we have $\frac{2+8+12+x}{4} = 10$, giving $x = 18$. In the second case, we have $\frac{x+2+8+12}{4} = 5$, giving $x = -2$. In the third case, we have $\frac{2+8+x+12}{4} = \frac{8+x}{2}$, yielding $22 + x = 16 + 2x$, or $x = 6$, making the desired value $18 + (-2) + 6 = 22$.

1-3 The geometric mean of n numbers is the n^{th} root of the product of the numbers. Set A contains three distinct positive integers. Set B contains three elements that are each exactly 30 more than a corresponding element in set A. Set C contains three elements that are each exactly 20 times a corresponding element in set A. The geometric mean of the sum of the elements of A and the sum of the elements of B is equal to the average (arithmetic mean) of the elements of sets A and C. What is the largest possible value of an element of set B?
[Answer: 35]

Let x be the sum of the elements of set A. The sum of the elements of set B is therefore $90 + x$ and the sum of the elements of set C is therefore $20x$. Setting up the relationship in the problem gives $\sqrt{x(90 + x)} = \frac{x+20x}{6} = \frac{7}{2}x$. Squaring both sides gives $x(90 + x) = \frac{49}{4}x^2$, or $\frac{45}{4}x - 90$, or $x = 8$. This means set A contains three distinct positive integers that add to 8. Therefore set A is either $\{1,2,5\}$ or $\{1,3,4\}$. Thus the largest possible value of an element of set A is 5, making the desired value 35.

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Round 2: Quadratic Equations

2-1 The functions $f(x) = x^2 - 8x + 15$ and $g(x) = x^2 + px - 36$, where p is a constant, both have integer zeros, one of which is common to both functions. Find the value of $f(p)$.
[Answer: 24]

Since $f(x)$ has zeros of $x = 3$ and $x = 5$, and one of those must be a factor of 36, we have that $x = 3$ must be common to both functions. Therefore $g(x) = (x - 3)(x + 12)$, giving $p = 9$, and $f(9) = 9^2 - 8(9) + 15 = 24$.

2-2 A quadratic function $f(x)$ has the same vertex as $g(x) = 2x^2 - 24x + 70$ and one zero of $6 - 2\sqrt{2}$. What is the y −coordinate of the y −intercept of $f(x)$?
[Answer: 7]

The vertex of $g(x)$ has an x-coordinate of $\frac{24}{2(2)} = 6$, making the coordinates $(6, -2)$. This means that $f(x) = a(x - 6)^2 - 2$. Using the provided point gives $0 = a(6 - 2\sqrt{2} - 6)^2 - 2$, or $8a - 2 = 0$, making $a = \frac{1}{4}$. Therefore the desired value is $\frac{1}{4}(0 - 6)^2 - 2 = 7$.

2-3 The quadratic equation $z^2 - (3 + pi)z + 60 = pi$, where p is a real constant and i is the imaginary constant, has one solution of $z = 3 + bi$ for some real number b. Find the value of b^2.
[Answer: 240]

The two solutions must have a sum of $3 + pi$ and a product of $60 - pi$. Since we are told one solution is of the form $3 + bi$, that means the second must be of the form ai where a is a real number so that the sum has a real component of 3, meaning $p = a + b$. The product of these solutions would be $-ab + 3ai$, making $60 = -ab$ and $-p = 3a$. This gives $-3a = a + b$, or $a = -\frac{1}{4}b$. Therefore $60 = -\left(-\frac{1}{4}b\right)b$, or $b^2 = 240$, which is our desired value.

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Round 3: Similarity

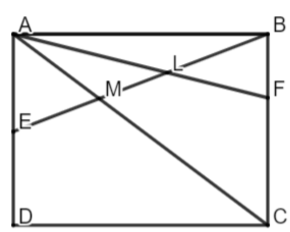
3-1 A polygon has a perimeter of 10 units and an area of 20 square units. A similar polygon has an area of 40 square units and a perimeter of d units. Find the area in square units of a square with diagonal length d.
[Answer: 100]

Since $\left(\frac{10}{d}\right)^2 = \frac{20}{40}$, we have $\frac{10}{d} = \frac{1}{\sqrt{2}}$, so $d = 10\sqrt{2}$. Therefore the square will have side length 10 and area 100, which is the desired value.

3-2 Refer to the diagram. A trough shaped like an isosceles triangular prism is filled completely with water. The rectangular top of the trough has dimensions of 12 inches by 20 inches, and the trough holds 2880 cubic inches of water. A slow leak causes the trough to lose water at a rate of 4 cubic inches per minute. If the leak continues, after how many minutes will the water in the trough have dropped by eight inches?
[Answer: 400]

Since the volume of the trough is $\frac{1}{2}(12)(20)h = 2880$, we know the height of the trough is 24 inches. After the water in the trough has dropped by 8 inches, the water will be in the shape of an isosceles triangular prism with a height $\frac{2}{3}$ of that of the initial prism but with the same depth. Therefore we can set up $\left(\frac{2}{3}\right)^2 = \frac{x}{2880}$ to solve for the new volume, giving $x = 1280$. This means that the trough lost $2880 - 1280 = 1600$ cubic inches of water, making the desired value 400.

3-3 Consider rectangle $ABCD$ (see the diagram, not necessarily to scale) with point E on $\overline{AD}$ such that $AE = ED$, and point F on $\overline{BC}$ such that $FC = 2BF$. $\overline{BE}$ intersects $\overline{AC}$ at point M and $\overline{BE}$ intersects $\overline{AF}$ at point L. The ratio of the area of $FCML$ to the area of $ABCD$ is $\frac{p}{q}$ where p and q are integers with no common factors greater than 1. Find $2q - p$.
[Answer: 26]



For simplicity, let $AE = ED = 3$, $BF = 2$, and $FC = 4$. Because the ratio of the length of the altitude of triangle AEM with vertex M to the length of the altitude of triangle BMC with vertex M is $1:2$ and the ratio fo the length of the altitude of triangle ALE from point L to the length of the altitude of triangle FLB from point L is $3:2$, for simplicity let $AB = 15$. Therefore the area of triangle CMB is $\frac{1}{2}(6)(10) = 30$, the area of triangle FLB is

$\frac{1}{2}(2)(6) = 6$, and the area of $ABCD$ is $(6)(15) = 90$. Therefore the ratio of the area of $FCML$ to the area of $ABCD$ is $\frac{30-6}{90} = \frac{4}{15}$, making the desired value $2(15) - 4 = 26$.

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Round 4: Variation

4-1 If y varies directly as the square of x and $y = 2$ when $x = \frac{1}{3}$, find the sum of the squares of all possible values of x when $y = 288$.
[Answer: 32]

Setting up $\frac{2}{\left(\frac{1}{3}\right)^2} = \frac{288}{x^2}$ yields $x^2 = 16$, or $x = \pm 4$, making the desired value $4^2 + (-4)^2 = 32$.

4-2 Assume that z varies directly with x and inversely as the square of y, and $z = 5$ when $x = 3$ and $y = 4$. For how many ordered pairs (x, y) where x and y are positive integers less than 150 is $z = 5$?
[Answer: 7]

We must count all ordered pairs (x, y) where $\frac{x}{y^2} = \frac{3}{4^2}$, so $x = \frac{3}{16}y^2$. Therefore y must be a multiple of 4. This yields the ordered pairs $(3,4), (12,8), (27,12), (48,16), (75,20), (108,24), (147,28)$, making the desired value 7.

4-3 Dr. Dootiny is studying animal populations in an isolated ecosystem. There are white mice and brown mice competing for resources and a species of lizard that only eats brown mice. She makes a model saying that the population of white mice is inversely proportional to the population of brown mice, and the population of lizards varies directly as the square root of the population of brown mice. In her first week, she counts 2000 white mice and 450 lizards. After 6 months the population of lizards had doubled and there were 14,400 brown mice. If her model is correct, the product of the populations of the two types of mice at any time is n thousand $(n * 10^3)$ for some integer n. Find the value of n.
[Answer: 7200]

Let b be the population of brown mice, w be the population of white mice, and L be the population of lizards. The model states that $w = \frac{p}{b}$ and $L = q\sqrt{b}$ for constants p and q. In the first week, we have $450 = q\sqrt{b} = q\sqrt{\frac{p}{2000}}$, and after six months we have $900 = q\sqrt{14400} = 120q$. Therefore $q = \frac{900}{120} = \frac{15}{2}$. Substituting this into the first equation yields $450 = \frac{15}{2}\sqrt{\frac{p}{2000}}$, or $\sqrt{\frac{p}{2000}} = 60$, so $p = 60^2 * 2000 = 7200 * 1000$, making the desired value 7200.

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Round 5: Trig Expressions & DeMoivre's Theorem

5-1 If A and B are angles in quadrant one such that $\cos(A) = \frac{5}{13}$ and $\cos(B) = \frac{3}{4}$, then $\sin(A+B) = \frac{a+b\sqrt{c}}{d}$ where a, b, c, and d are integers, c has no perfect square factors greater than 1, and a, b, and d have no common factors greater than 1. Find the value of $a + 2b + 3c - d$.
[Answer: 15]

Since $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$, the value is $\left(\frac{12}{13}\right)\left(\frac{3}{4}\right) + \left(\frac{5}{13}\right)\left(\frac{\sqrt{7}}{4}\right) = \frac{36+5\sqrt{7}}{52}$, making the desired quantity $36 + 2(5) + 3(7) - 52 = 15$.

5-2 One of the complex twelfth roots of a complex number z has an argument of $\frac{13}{10}\pi$. If each of the twelfth roots of z is written with an argument between 0 and 2π, then the smallest of these arguments is $\frac{a}{b}\pi$, where a and b are positive integers with no common factors greater than 1. The argument of z is $\frac{c}{d}\pi$, where c and d are positive integers with no common factors greater than 1 and $0 \leq \frac{c}{d} < 2$. Find the value of $a + 2b + 3c + 4d$.
[Answer: 76]

To find the smallest positive argument of a complex twelfth root of z, we subtract multiples of $\frac{2\pi}{12}$ from $\frac{13\pi}{10}$ until we reach the smallest positive result: $\frac{39\pi}{30} - 7\left(\frac{5\pi}{30}\right) = \frac{4\pi}{30} = \frac{2}{15}\pi$. The argument of z is $12\left(\frac{13\pi}{10}\right) = \frac{78}{5}\pi$, which is equivalent to an argument of $\frac{8}{5}\pi$. Therefore the desired quantity is $2 + 2(15) + 3(8) + 4(5) = 76$.

5-3 If $\tan\left(x + \frac{\pi}{3}\right) = 5\sqrt{3}$, then $\sec^2(x) = \frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $a + 2b$.
[Answer: 51]

Using the $\tan(A+B)$ formula, we have $\frac{\tan(x)+\sqrt{3}}{1-\sqrt{3}\tan(x)} = 5\sqrt{3}$, yielding $\tan(x) + \sqrt{3} = 5\sqrt{3} - 15\tan(x)$, giving $\tan(x) = \frac{\sqrt{3}}{4}$. Therefore $\sec^2(x) = 1 + \frac{3}{16} = \frac{19}{16}$, making the desired quantity $19 + 2(16) = 51$.

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Round 6: Conic Sections

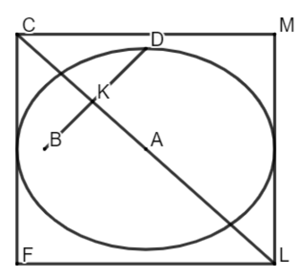
6-1 A particular conic section is the set of all points equidistant from the point $(8,2)$ and the line $x = 6$. What is the $x-$value of the point on this conic where $y = 8$?
[Answer: 16]

The conic section making up the set of all points equidistant from a point and a line is a parabola. This parabola opens horizontally in the direction of the positive x-axis with a vertex of $(7,2)$ and the distance between the focus and vertex is 1. Therefore the equation is $x = \frac{1}{4}(y-2)^2 + 7$, and substituting $y = 8$ yields $x = \frac{1}{4}(8-2)^2 + 7 = 16$.

6-2 A circle has a center that lies on $x = -4$, has a radius of 3, and contains the point $(-5,10)$. Find the product of all possible $y-$coordinates of the center of this circle.
[Answer: 92]

The distance between the x-coordinates of the center and the point $(-5,10)$ is 1, making the distance between the y-coordiantes of the center and the point $(-5,10)$ is $\sqrt{9-1} = 2\sqrt{2}$. Therefore the $y-$coordinate of the center must be $10 + 2\sqrt{2}$ or $10 - 2\sqrt{2}$, making the product $100 - 8 = 92$.

6-3 Refer to the diagram (not necessarily drawn to scale) that shows an ellipse with the same center as a rectangle FCML where $\overline{FC}$ and $\overline{ML}$ are tangent to the ellipse. The ellipse has a major axis of length 18 and a minor axis of length 14. The center of the ellipse is point A and one of the foci is labeled point B, so $\overline{FL} || \overline{AB}$. The point on the ellipse with the largest y-value is point D. A diagonal of the rectangle intersects $\overline{BD}$ at point K so that $BK = AK$. The area of the rectangle is $\frac{a\sqrt{b}}{c}$ where b has no perfect square factors greater than 1 and a and c have no common factors greater than 1. Find $a - b + c$.
[Answer: 567]



Let P be the midpoint of $\overline{FC}$. Note that from the properties of ellipses, $BD = AP = 9$, $AD = 7$, and $AB = \sqrt{81-49} = 4\sqrt{2}$. Because $BK = AK$, $\angle KBA \cong \angle KAB$, and triangle CAP is similar to triangle DBA. Therefore $\frac{PC}{AP} = \frac{AD}{BA}$, giving $\frac{PC}{9} = \frac{7}{4\sqrt{2}}$, and so $PC = \frac{63\sqrt{2}}{8}$. This means $FC = \frac{63\sqrt{2}}{4}$, and the area of $FCML$ is $(18)\left(\frac{63\sqrt{2}}{4}\right) = \frac{567\sqrt{2}}{2}$, making the desired quantity $567 - 2 + 2 = 567$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 4
Team Round

Please write your answers on the answer sheet provided.

1. A set of three distinct integers has the property that doubling the largest number would increase the range by 40% and increase the average (arithmetic mean) by 12. What is the positive difference between the largest and smallest possible values of the median of the original set?
[Answer: 88]

Let the three integers be called $a, b,$ and c, with $a > b > c$. We therefore have $2a - c = 1.4(a - c)$, and $\frac{2a+b+c}{3} = \frac{a+b+c}{3} + 12$. The first equation yields $.6a = -.4c$, or $a = -\frac{2}{3}c$. The second equation yields $2a = a + 36$, or $a = 36$, making $c = -54$. This means the median of the set, b, has a maximum possible value of 35 and a minimum possible value of -53, making the desired value $35 - (-53) = 88$.

2. There are two values of the constant m such that the equation $\frac{21}{m^3}x^2 + \frac{10}{m}x = \frac{7-4m}{m}x - \frac{9m}{4} - \frac{10}{m^2}x^2$ has exactly one real solution for x. If the two values of m are m_1 and m_2 such that $m_1 > m_2$, find the value of $-\frac{5m_1}{m_2}$.
[Answer: 16]

Writing this quadratic in standard form yields $\frac{21+10m}{m^3}x^2 + \frac{3+4m}{m}x + \frac{9m}{4} = 0$. Multiplying each term by $4m^3$ gives $(84 + 40m)x^2 + (12m^2 + 16m^3)x + 9m^4 = 0$. Since there is only one real solution for x, it follows that $(12m^2 + 16m^3)^2 - 4(9m^4)(84 + 40m) = 0$. This expression can be divided by $16m^4$ (since $m = 0$ is extraneous) to make $(3 + 4m)^2 - 9(21 + 10m) = 0$, and expanded out this makes $8m^2 - 33m - 90 = 0$, which factors into $(m - 6)(8m + 15) = 0$. Therefore $m_1 = 6$ and $m_2 = -\frac{15}{8}$, making the desired value $\frac{5(6)(8)}{15} = 16$.

3. A particular irregular prism has a height of 10 units, a surface area of 25 square units, and a volume of 40 cubic units. For how many different integer heights h such that $10 < h < 100$ would a similar prism have an integer surface area in square units or an integer volume in cubic units?
[Answer: 53]

Let a potential surface area of a similar prism be A and a potential volume of a similar prism be V. We have $\left(\frac{h}{10}\right)^2 = \frac{A}{25}$, so $A = \frac{1}{4}h^2$, and $\left(\frac{h}{10}\right)^3 = \frac{V}{40}$, making $V = \frac{1}{25}h^3$. Therefore, for either A or V to be an integer, h must be a multiple of 2 or 5. For $10 < h < 100$, there are 44 multiples of 2, 17 multiples of 5, and 8 multiples of 10, making the desired quantity $44 + 17 - 8 = 53$.

4. For some positive number n, z varies directly as x to the power of n and inversely as y to the power of $n + 1$. If $z = \frac{1}{16}$ when $x = 1$ and $y = 16$, and $z = 16$ when $x = 256$ and $y = 1$, find z when $x = 128$ and $y = 4$.
[Answer: 2]

Setting up $\frac{\left(\frac{1}{16}\right)*16^{n+1}}{1^n} = \frac{16*1^{n+1}}{256^n}$, we have $256^n * 16^n = 16$, or $16^{3n} = 16$, making $n = \frac{1}{3}$. Therefore we have $16^{1/3} = \frac{z*4^{4/3}}{128^{1/3}}$, so the desired value is $z = \frac{128^{1/3}*16^{1/3}}{4^{4/3}} = \frac{2^{7/3}*2^{4/3}}{2^{8/3}} = 2$.

5. If $\cos\left(x - \frac{\pi}{4}\right) = \frac{7}{8}$, find the value of $32\sin(2x)$.
[Answer: 17]

Using the $\cos(A - B)$ formula, we have $\frac{\sqrt{2}}{2}\cos(x) + \frac{\sqrt{2}}{2}\sin(x) = \frac{7}{8}$, or $\cos(x) + \sin(x) = \frac{7\sqrt{2}}{8}$. Squaring both sides makes $1 + 2\sin(x)\cos(x) = \frac{98}{64} = \frac{49}{32}$, yielding $\sin(2x) = \frac{17}{32}$, and therefore $32\sin(2x) = 17$.

6. The eccentricity of a hyperbola is defined as the ratio of the distance between the center and one focus to the distance between the center and one vertex. A hyperbola with equation $Ax^2 + By^2 + Cx + Dy + E = 0$ for real constants $A, B, C, D,$ and E has an eccentricity of 2.6, one focus at $(5,2)$, and another focus in quadrant II 14 units away horizontally from the first focus. If the equation of the asymptote with a positive slope is $f(x)$, find $f(13)$.
[Answer: 38]

The second focus must be at $(-9,2)$, making the center of the hyperbola $(-2,2)$. If the hyperbola has an equation of the form $\frac{(x+2)^2}{a^2} - \frac{(y-2)^2}{b^2} = 1$, then applying the eccentricity, we get $\frac{13}{5} = \frac{\sqrt{a^2+b^2}}{a}$, and squaring both sides yields $\frac{169}{25} = \frac{a^2+b^2}{a^2} = 1 + \frac{b^2}{a^2}$, so $\left(\frac{b}{a}\right)^2 = \frac{144}{25}$, so $\frac{b}{a} = \pm\frac{12}{5}$. Therefore the positive asymptote slope is $\frac{12}{5}$, giving $f(x) = \frac{12}{5}(x + 2) + 2$, and so $f(13) = \frac{12}{5}(13 + 2) + 2 = 38$.