FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 3
Individual Section

Please write your answers on the answer sheet provided.

Round 1: Decimals and Base Notation

1-1 Let S be the set of 17 numbers that can be written as 31_b where b is an integer and $4 \leq b \leq 20$. The set S contains p prime numbers and n perfect square numbers. Find the value of $2p + 3n$.
[Answer: 21]

The set S will contain (in base 10) the numbers $\{13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 52, 55, 58, 61\}$. The numbers $13, 19, 29, 31, 37, 53$, and 61 are prime and the numbers $16, 25$, and 49 are perfect squares. Therefore $p = 6$ and $n = 3$, making the desired quantity $2(6) + 3(3) = 21$.

1-2 If $\frac{(5*10^3)^x(2^{3x})}{(2*10^6)^{x-2}} = 10^n$ for integers x and n, what is the value of n?
[Answer: 16]

Applying all exponents in the expression on the left gives $\frac{5^x*10^{3x}*2^{3x}}{2^{x-2}*10^{6x-12}}$, and if we apply the division we get $5^x * 2^{2x+2} * 10^{-3x+1}$. Since this expression must equal an integer power of 10, we can conclude that the exponents of 5 and 2 are the same, so $x = 2x + 2$, This gives $x = -2$, making the expression $5^{-2} * 2^{-2} * 10^{18}$, or 10^{18-2}, so $n = 16$.

1-3 For some integer base x, $210_x - 101_{x+1} = 202_{x-4}$. Express the value of 10_x as a numeral in base 10.
[Answer: 12]

Turning this relationship into an equation yields $2x^2 + x - ((x + 1)^2 + 1) = 2(x - 4)^2 + 2$. Expanding the binomials and writing as a quadratic in standard form yields $x^2 - 15x + 36 = 0$, which gives possible values of $x = 12$ and $x = 3$. However, $x = 3$ gives one number in the equation a base of -1, making this an extraneous solution. Therefore, $x = 12$, which is also the desired answer.

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Round 2: Word Problems

2-1 Mr. Hill brought some money to the local bookstore. He spent $\frac{1}{3}$ of the money he brought on a science book and $\frac{2}{3}$ of the remaining money he brought on a history book. If the difference between the prices of the books was $8.75, what was the cost in dollars of the history book?
[Answer: 35]

The science book cost Mr. Hill $\frac{1}{3}$ of his money and the history book cost him $\frac{2}{3}\left(\frac{2}{3}\right)$ or $\frac{4}{9}$ of his money. Since $\frac{4}{9}-\frac{1}{3}=\frac{1}{9}$ and this fraction of his money was $8.75, this means the history book cost $4(\$8.75)=\$35$, making our answer 35.

2-2 Mr. Allwood and Mr. Krupnikoff are raising racing slugs. They decide to race their slugs down a 36-inch yard stick. Mr. Allwood bets Mr. Krupnikoff that his slug is so fast his could win even if Mr. Krupnikoff's had a five minute head start. Mr. Krupnikoff's slug starts down the yard stick at a pace of 3 inches per minute, and five minutes later Mr. Allwood's slug starts down the yard stick at a pace of 5 inches per minute. How many inches apart are the two slugs when the first one crosses the finish line?
[Answer: 1]

At a rate of 3 inches per minute, Mr. Krupnikoff's snail would finish the race in 12 minutes. At this time, Mr. Allwood's snail would have been racing for 7 minutes, but that means it would only have traveled 35 inches. This means that Mr. Krupnikoff's snail crosses the finish line first and the distance between the two slugs is 1 inch, making 1 the desired answer.

2-3 A garbage can in a public park is put out in the morning every day, and fills three times faster from 9:00 AM on than it does before 9:00 AM. The fill rate every day prior to 9:00 AM is the same. On Monday, by the time the can was full, it had been out twice as long from 9:00 AM on as it had been prior to 9:00 AM. On Tuesday, the can is put out 45 minutes later than it was on Monday (though still prior to 9:00 AM) and was full by 12:05 PM. What time was the can put out on Monday morning? Enter your answer as a three-digit integer with no colon (for example: enter the time 6:17 as 617).
[Answer: 735]

Let x be the number of minutes the can is out prior to 9:00 AM, and let r represent the morning fill rate. Since the can's full amount is the same both days and (fill rate)(fill time) = fill amount, we can create an equation setting Monday equal to Tuesday: $rx+(3r)(2x)=r(x-45)+185(3r)$. We can eliminate r from the equation to get $7x=x-45+555$, or $6x=510$, making $x=85$. This means the can was put out 85 minutes prior to 9 on Monday, or $7{:}35$ AM. This makes our desired value 735.

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Round 3: Polygons

3-1 For how many values of $n < 100$ are the number of diagonals of the n −gon a positive integer multiple of n?
[Answer: 48]

Since the number of diagonals of an n-gon is $\frac{n(n-3)}{2}$, this will be a positive integer multiple of n if $\frac{n-3}{2}$ is a positive integer. This will be true of any odd integer greater than 3, so all odd integers from 5 to 99 work, making 48 the desired value.

3-2 A regular n-gon has the property that one of its interior angles measures exactly ten degrees more than the measure of one interior angle of a regular 28-gon. What is the value of n?
[Answer: 126]

From the description, we have $180 - \frac{360}{n} = 10 + 180 - \frac{360}{28}$, which gives $\frac{360}{n} = \frac{80}{28} = \frac{20}{7}$, so $n = \frac{360*7}{20} = 126$.

3-3 Let n_1 be the number of sides of a polygon with the property that the number of diagonals is less than the number of diagonals of an $(n_1 + 2)$-gon by the measure in degrees of an exterior angle of a regular octagon. Let n_2 be the number of sides of a regular polygon with the property that it is the only regular polygon whose interior angle measures in degrees is a prime integer. Find the value of $2n_1 + n_2$.
[Answer: 406]

We can solve for n_1 given $\frac{n_1(n_1-3)}{2} + 45 = \frac{(n_1+2)(n_1-1)}{2}$, so $n_1^2 - 3n_1 + 90 = n_1^2 + n_1 - 2$, we gives $4n_1 = 92$ so $n_1 = 23$. We can determine the value of n_2 knowing that the interior angle measure will be $180 - \frac{360}{n_2}$, and since $\frac{360}{n_2}$ can take values of $\{1,2,3,4,5,6,8,9,10,12,15,18,20,24,30,36,40,45,60,72,90,120\}$, the only one that produces a prime result when subtracted from 180 is 1, making $n_2 = 360$. Therefore the desired value is $2(23) + 360 = 406$.

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Round 4: Function and Inverses

Note: the inverse f^{-1} of a function is not necessarily a function.

4-1 If $f(x) = 4x - \frac{8}{5}$, find $f^{-1}(f^{-1}(88))$.
[Answer: 6]

Solving for the inverse of f yields $f^{-1}(x) = \frac{1}{4}x + \frac{2}{5}$, and $f^{-1}(f^{-1}(x)) = \frac{1}{4}\left(\frac{1}{4}x + \frac{2}{5}\right) + \frac{2}{5} = \frac{1}{16}x + \frac{1}{2}$, making the desired value $\frac{1}{16}(88) + \frac{1}{2} = 6$.

4-2 Let $f(x)$ be a one-to-one function that has domain $[-12,8]$ and range $[1,10]$. If $g(x) = 3f(2x+6) - 7$, then $g(x)$ has a domain $[a, b]$ and a range of $[c, d]$, where a, b, c, and d are integers. Find the value of $b - 2a + d - 2c$.
[Answer: 50]

Setting up $2a + 6 = -12$ gives $a = -9$, and likewise setting up $2b + 6 = 8$ gives $b = 1$, making the domain of $g(x)$ $[-9,1]$. The minimum of the range of f is 1, so $c = 3(1) - 7 = -4$. The maximum of the range of f is 10, so $d = 3(10) - 7 = 23$, making the range $[-4,23]$. The desired value is then $1 - 2(-9) + 23 - 2(-4) = 50$.

4-3 Consider functions $f(x) = \sqrt{x+2}$ and $g(x) = \frac{1}{x^2-9}$ and real numbers c_1, c_2, and c_3. c_1 is in the domain of f but not in the domain of g. c_2 is in the domains of neither f nor g. c_3 is in the domains of both f and g but NOT in the domain of $g \circ f$. Find the value of $f^{-1}(c_1 + 2c_2 + 3c_3)$.
[Answer: 322]

The domain of f is $[-2, \infty)$ and the domain of g is $(-\infty, -3) \cup (-3,3) \cup (3, \infty)$. Therefore, $c_1 = 3$ and $c_2 = -3$. For all $x \geq -2$, $g \circ f = \frac{1}{x-7}$, so $c_3 = 7$. Therefore, the desired value is x such that $\sqrt{x+2} = 3 + 2(-3) + 3(7) = 18$, so $x = 322$.

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Round 5: Exponents & Logarithms

5-1 If $x = \log_7 14$ and $y = \log_7 245$, find the value of 7^{x+y-2}.
[Answer: 70]

Note that $x + y - 2$ is $\log_7\left(\frac{14*245}{49}\right) = \log_7(70)$, so $7^{x+y-2} = 70$.

5-2 If $(3^b)^{b-a} = 9^{\frac{1}{2}a^2-\frac{1}{2}ab-2}$ and $a + b = 12$, find the value of 64^{a-b}.
[Answer: 4]

Applying the exponent on the left hand side and rewriting the right hand side in terms of base 3 gives $3^{b^2-ab} = 3^{a^2-ab-}$, and setting $b^2 - ab = a^2 - ab - 4$ gives $a^2 - b^2 = 4$. Given $a + b = 12$, this means $4 = (a + b)(a - b) = 12(a - b)$, so $a - b = \frac{1}{3}$, so $64^{\frac{1}{3}} = 4$.

5-3 If $6\log_3 x + 11 = 10\log_x 3$ and $x > 1$, find the value of $x^3 - 1$.
[Answer: 8]

Note that the right hand side can be rewritten using the change of base property so that $6\log_3 x + 11 = 10\left(\frac{\log_3 3}{\log_3 x}\right) = \frac{10}{\log_3 x}$. Letting $u = \log_3 x$, we get the quadratic $6u^2 + 11u - 10 = 0$. We can solve by factoring $(2u + 5)(3u - 2) = 0$. This gives solutions of $\log_3 x = -\frac{5}{2}$, or $x = 3^{-5/2}$, and $\log_3 x = \frac{2}{3}$, or $x = 3^{2/3}$. Since only the second option is greater than one, the desired quantity is $\left(3^{2/3}\right)^3 - 1$ or 8.

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Round 6: Matrices

6-1 If $\begin{bmatrix}5\\x\end{bmatrix}+\begin{bmatrix}2 & -1\\3 & y\end{bmatrix}\begin{bmatrix}x\\6\end{bmatrix}=\begin{bmatrix}13\\88\end{bmatrix}$, what is the value of y?
[Answer: 10]

Rewriting the top row as an equation gives $5+2x-6=13$, which yields $x=7$. Writing the second row as an equation gives $x+3x+6y=88$, and since $x=7$, we have $28+6y=88$, which leads to the desired value of $y=10$.

6-2 Consider matrices $A=\begin{bmatrix}x & y\\-y & x\end{bmatrix}$ and $B=\begin{bmatrix}3x & y\\x & 3y\end{bmatrix}$. If $det(A)=18$ and $det(B)=28$, find the square of the sum of the elements of B.
[Answer: 400]

Using the determinant of A, we have $x^2+y^2=18$. Using the determinant of B, we have $9xy-xy=8xy=28$. The square of the sum of the elements of B is $(4x+4y)^2=16(x+y)^2$. Since $2xy=7$, we have $x^2+2xy+y^2=25$, or $(x+y)^2=25$, so the desired value is $(16)(25)=400$.

6-3 There are two values of x which make the matrix $\begin{bmatrix}6 & -1 & x\\x-1 & 4 & 14\\-x & 1 & 0\end{bmatrix}$ singular (non-invertible). If the two values are x_1 and x_2 and $x_1>x_2$, find the value of $3x_1-5x_2$.
[Answer: 37]

The determinant of the matrix in terms of x is $6(-14)+14x+x(x-1+4x)$, and writing this in standard form and setting equal to 0 (because the matrix is singular) yields $5x^2+13x-84=0$. This factors into $(x-3)(5x+28)=0$, giving $x_1=3$ and $x_2=-\frac{28}{5}$, so the desired value is $3(3)-5\left(-\frac{28}{5}\right)=37$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 3
Team Round

Please write your answers on the answer sheet provided.

1. For how many integer bases $p > 6$ is the number $14641_p \leq 1.3 * 10^7$?
[Answer: 53]

Notice that the left side of the inequality translates to $p^4 + 4p^3 + 6p^2 + 4p + 1$, which is equivalent to $(p+1)^4$. Taking the fourth root of both sides of the inequality yields $p + 1 \leq 10\sqrt[4]{1300}$. The right side is only slightly larger than 10(6), so we can write the inequality to solve for integer values $p + 1 \leq 60$, making $p \leq 59$. Given the restriction of p in the problem, we have $7 \leq p \leq 59$, making for 53 possible values.

2. Randy and Katina have one hour to finish a huge pizza as part of a local challenge. On this day, Randy's eating rate is twice that of Katina's. After they have finished one third of the pizza, Randy needs to take a 20 minute break, after which he resumes eating and they finish the pizza together after another 24 minutes. When they finished the pizza, how many *seconds* were left until time was up?
[Answer: 40]

If r is Randy's rate in pizzas per minute and k is Katina's rate in pizzas per minute, we can set up the problem given the information as $\frac{1}{3} + 20k + 24(k + r) = 1$. Since $r = 2k$, we can write and write this as $20k + 24(3k) = \frac{2}{3}$. Solving for k gives $k = \frac{1}{138}$, and so $r = \frac{1}{69}$ and their combined rate is $\frac{1}{46}$. To find the time m in minutes that it took to finish the first third, we solve $\frac{m}{46} = \frac{1}{3}$, so $m = 15\frac{1}{3}$, or 15 minutes and 20 seconds. Therefore they spent a total of 59 minutes and 20 seconds eating the pizza, so the desired number of seconds left until time was up was 40.

3. Consider regular 20-gon $ABC \ldots. RST$. The angles formed by every diagonal with one endpoint A and $\overline{AB}$ are measured. What is the total sum in degrees of the measures of these angles?
[Answer: 1377]

Since the measure of one interior angle of the 20-gon is $180 - \frac{360}{20} = 162°$, and triangle ABC is isosceles with congruent sides $\overline{AB}$ and $\overline{BC}$, it follows that $m\angle BAC = 9°$. We can use a similar process to find that $m\angle BAD = 18°\ \left(\frac{2(180)-2(162)}{2}\right)$. Algebraically we can determine that the measure of each successive angle is 9° more than the previous one. This makes the total sum in degrees $9 + 18 + \cdots + 9(17)$. This sum can be computed quickly doing $9(1 + 2 + \cdots + 17) = 9(9)(17) = 1377$.

4. Let $f(x) = \frac{x}{\sqrt{x+3}-1}$ and $g(x) = \left|f(x) - \sqrt{x+3} - 1\right|$. If a is the smallest integer such that $g(x) \leq .01$ for all $x \geq a$, what is the value of a?
[Answer: 40398]

Note that $g(x)$ is the positive difference between $f(x)$ and the quantity $\sqrt{x+3} + 1$. Inspection can note that for large values of x (e.g. $x = 97$) the quantity $\sqrt{x+3} + 1$ is larger than $f(x)$. Setting up $\sqrt{x+3} + 1 - \frac{x}{\sqrt{x+3}-1} \leq .01$ and then multiplying through by $\sqrt{x+3} - 1$ (presumed positive for large values of x) yields $x + 2 - x \leq .01(\sqrt{x+3} - 1)$, or $200 \leq \sqrt{x+3} - 1$. Solving for x yields $x \geq (201)^2 - 3$, or $x \geq 40401 - 3 = 40398$, which is our desired value.

5. Solve for the value of N:

$$\sum_{k=1}^{N} \log\left(\frac{k}{k+1}\right) + \prod_{k=N+1}^{9999} \log_k(k+1) = 0$$

Note: $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$ and $\prod_{k=1}^{n} a_k = a_1 a_2 a_3 \ldots a_n$
[Answer: 99]

We note first that $\log\left(\frac{1}{2}\right) + \log\left(\frac{2}{3}\right) + \cdots + \log\left(\frac{N}{N+1}\right)$ is equivalent to $\log\left(\frac{1*2*\ldots*N}{2*3*\ldots*(N+1)}\right)$, which simplifies to $\log\left(\frac{1}{N+1}\right)$. Next we note that $\log_{N+1}(N+2) * \log_{N+2}(N+3) * \ldots * \log_{9999}(10000)$ simplifies using change of base to $\log_{N+1}(10000)$. So our equation becomes $\log\left(\frac{1}{N+1}\right) + \log_{N+1}(10000) = 0$, which can also be written as $\log_{N+1}(10000) = \log(N+1)$. Using change of base, we have $\frac{\log(10000)}{\log(N+1)} = \log(N+1)$, or $(\log(N+1))^2 = 4$, or $\log(N+1) = \pm 2$. Since N must be an integer, the value of -2 is extraneous. Therefore $N + 1 = 10^2$, or $N = 99$.

6. Consider matrices $A = \begin{bmatrix} 12 & a \\ 4 & b \end{bmatrix}$ and $B = \begin{bmatrix} c & -8 \\ -1 & d \end{bmatrix}$ where $a, b, c,$ and d are constants. If $A^{-1} = B$, find the determinant of the matrix $\begin{bmatrix} a & d \\ b & c \end{bmatrix}$.
[Answer: 55]

Using the inverse formula, we know that $A^{-1} = \frac{1}{det(A)} \begin{bmatrix} b & -a \\ -4 & 12 \end{bmatrix}$. Therefore $\frac{-4}{det(A)} = -1$, so $det(A) = 4$. This also means that $\frac{-a}{4} = -8$, so $a = 32$, and $\frac{12}{4} = d = 3$. We can solve for b since $12b - (32)(4) = 4$, making $b = 11$. Finally we know that $c = \frac{b}{4} = \frac{11}{4}$. This makes the desired quantity $(32)\left(\frac{11}{4}\right) - (3)(11) = 55$.