FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 2

Round 1: Factors and Multiples

1-1 If an integer is selected from the set of odd integers greater than 1 and less than 50, then the probability it is prime is $\frac{a}{b}$ where a and b are positive integers with no common factors greater than 1. Find $b - a$.
[Answer: 11]

Looking at the set of 24 odd integers greater than 1 and less than 50, we note the primes: {3,5,7,11,13,17,19,23,29,31,37,41,43,47}. There are 13 total, making the desired probability $\frac{13}{24}$, so $24 - 13 = 11$.

1-2 The greatest common factor between x and 180 is 15. The least common multiple of x and 225 is 1575. Find the sum of all possible values of x.
[Answer: 630]

Since $180 = 2^2 * 3^2 * 5$, we know x must have exactly one factor of 3, at least one factor of 5, and no factors of 2. Then, noting that $225 = 3^2 * 5^2$ and $1575 = 3^2 * 5^2 * 7$, we know x must have a factor of 7, but could have one or two factors of 5. This gives two possible values of x: 105 $(3 * 5 * 7)$ or 525 $(3 * 5^2 * 7)$. Therefore, the sum of all possible values is $105 + 525 = 630$.

1-3 For two positive integers p and q, the product pq is the smallest whole number with exactly 14 factors, and the greatest common factor of p and q is 4. Find the least common multiple of p and q.
[Answer: 48]

To have exactly 14 factors, the value pq must have a prime factorization of either a^{13} or a^6b, where a and b are distinct prime numbers. The smallest possible value would be $2^6 * 3$, or 192. Since $pq = lcm(p,q) * gcf(p,q)$ and $gcf(p,q) = 4$, the desired quantity is $\frac{192}{4} = 48$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 2

Round 2: Polynomials and Factoring

2-1 If $(2x+3)(ax^2+7x+b) = 10x^3+cx^2+13x+d$ for integers $a, b, c,$ and d, find $a+b+c+d$.
[Answer: 18]

Expanding the product gives $2ax^3+(14+3a)x^2+(2b+21)x+3b = 10x^3+cx^2+13x+d$. Equating the cubic terms gives $a=5$, which subsequently gives $c=29$ from the quadratic terms. Equating the linear terms then gives $b=-4$, which finally makes $d=-12$. The sum of the four numbers is $5+29-4-12=18$.

2-2 For how many distinct integer values of M is $27x^2+Mx+20$ factorable into two binomials with integer coefficients?
[Answer: 24]

Noting that the product of $27*20$ would have a prime factorization of 2^2*3^3*5, we know this product will have $3*4*2=24$ distinct factors, broken into 12 distinct pairs. Since M could be either the positive or negative sum of each pair of numbers, this leaves us with $2*12=24$ possible values of M.

2-3 The polynomial $f(x)=2x^3-24x^2+px-q$, where p and q are integers, has one zero of $x=1+\frac{3}{2}i$. What is the value of q?
[Answer: 65]

Because all coefficients of $f(x)$ are real, there must be another zero of $x=1-\frac{3}{2}i$. We also know that the sum of the zeros of $f(x)$ must be $-\frac{-24}{2}=12$, and since the two already known zeros add to 2, the remaining zero must be 10. Finally, given that the product of the zeros must be $-\frac{-q}{2}$ and the product is $10\left(1+\frac{3}{2}i\right)\left(1-\frac{3}{2}i\right)=10\left(1+\frac{9}{4}\right)=\frac{65}{2}$, it follows that $q=65$.

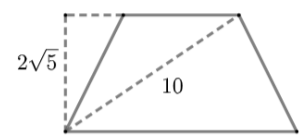
FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 2

Round 3: Area and Perimeter

3-1 A rectangle with integer dimensions has twice the area of a square with a perimeter of 24 units. What is the smallest possible perimeter of the rectangle?
[Answer: 34]

A square with a perimeter of 24 units would have a side length of 6 units and therefore an area of 36 square units. This means the rectangle with have an area of 72 square units. The smallest possible perimeter would occur when the rectangle's dimensions are as close as possible, giving dimensions of 8 units and 9 units, or a total perimeter of 34 units.

3-2 An isosceles trapezoid has a diagonal of length 10 and a height of $2\sqrt{5}$. What is the area of the trapezoid?
[Answer: 40]



See the diagram. We can construct a right triangle and find the length of the missing leg using the Pythagorean theorem, giving us $\sqrt{100-20} = 4\sqrt{5}$. Because the trapezoid is isosceles, this leg is also the length of the midsegment, or the average of the lengths of the short and long bases. Therefore, the area is $(2\sqrt{5})(4\sqrt{5}) = 40$.

3-3 Consider regular octagon $ABCDEFGH$. If the trapezoid $ABCD$ is reflected across its longer base, the resulting equilateral hexagon has area M. A regular hexagon with the same perimeter as the equilateral hexagon has area N. The ratio $\frac{M}{N}$ can be written in simplest radical form of $\frac{a\sqrt{b}+c\sqrt{d}}{f}$ where $a, b, c, d,$ and f are positive integers, b and d have no perfect square factors greater than 1, and $a, c,$ and f have no factors common to all of them greater than 1. Find $2(a+c)+3(b+d)-f$.
[Answer: 26]

The trapezoid $ABCD$ is an isosceles trapezoid with three congruent sides, including the smaller base. For simplicity assume the congruent sides all have length 1, giving the equilateral hexagon a perimeter of 6. The height of trapezoid is $\frac{1}{\sqrt{2}}$ and the bases have length 1 and $1+\sqrt{2}$. The area of the hexagon is twice the area of the trapezoid, or $2\left(\frac{1}{\sqrt{2}}\right)\left(1+\frac{\sqrt{2}}{2}\right) = 1+\sqrt{2}$. The regular hexagon would have an area of $6(1)\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$. The ratio of the areas is $\frac{1+\sqrt{2}}{\frac{3\sqrt{3}}{2}} = \frac{2+2\sqrt{2}}{3\sqrt{3}} = \frac{2\sqrt{3}+2\sqrt{6}}{9}$, making the desired quantity $2(2+2)+3(3+6)-9 = 26$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 2

Round 4: Absolute Value & Inequalities

4-1 The inequality $|x+7|<22$ has a solution set of $p<x<q$. Find $2q-3p$.
[Answer: 117]

Noting that $p=-7-22=-29$ and $q=-7+22=15$, the desired quantity is $2(15)-3(-29)=117$.

4-2 Consider the equation $|2x-a|=b$, where *a* and *b* are constants, and $b>0$, and suppose that the value of *b* is given. Then there are two values of *a* such that $x=18$ is a solution of the equation. Let these two values of *a* be a_1 and a_2. The sum of the other solution of the equation when $a=a_1$ and the other solution of the equation when $a=a_2$ is *k*, where *k* is independent of the initial choice of *b*. Find *k*.
[Answer: 36]

The equation is equivalent to $\left|x-\frac{a}{2}\right|=\frac{b}{2}$. If $18>\frac{a}{2}$, then the second solution lies $2\left(\frac{b}{2}\right)=b$ units below 18. However, if $18<\frac{a}{2}$, then the second solution lies b units above 18. Regardless of the values of a and b, the two possible values of the second solution lie symmetrically about 18, making 18 their midpoint. Therefore, the two values must sum to 36.

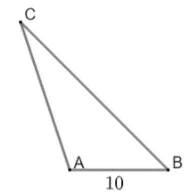
4-3 How many integers satisfy the inequality $\left|6-\frac{3}{4}x\right|-\left|\frac{1}{3}x+\frac{2}{3}\right|<4$?
[Answer: 24]

One way to approach this is graphically. Rewriting the inequality as $\left|6-\frac{3}{4}x\right|<\left|\frac{1}{3}x+\frac{2}{3}\right|+4$ and noting that this is equivalent to $|72-9x|<|4x+8|+48$, this can be approached as considering for what values of x the graph of $|4x+8|+48$, an absolute value graph with a vertex at $(-2,48)$, lies above the graph of $|72-9x|$, an absolute value graph with a vertex at $(8,0)$. We can confirm that at $x=-2$, the second graph lies above the vertex of the first, so the part of the graph to the left of the vertex of the second graph must be bounded above buy the part of the graph to the right of the vertex of the first. We can find the x-coordinate of the intersection point by setting $72-9x=4x+56$, giving $x=\frac{16}{13}$, which is greater than 1 but less than 2. The x −coordinate of the other point of intersection can be found by setting $9x-72=4x+56$, giving $x=\frac{128}{5}$, which is greater than 25 but less than 26, This means that all integers in the interval $(1,26)$ satisfy the inequality, giving 24 different values.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 2

Round 5: Law of Sines and Cosines

5-1 The Dutchman walks along a pier, and he spots a tugboat in the water 45 degrees north of west of his position. He walks 10 meters directly west along the pier and notices the tugboat is now 75 degrees north of west of his position. How far is the tugboat from his position in meters? Round your answer to the nearest whole meter.
[Answer: 14]



Refer to the diagram. We can model this problem using triangle ABC, presuming that the Dutchman starts at point B and walks west to point A with the tugboat being located at point C. We know the measure of angle B is 45 degrees and the measure of angle A is 105 degrees, making the measure of angle C 30 degrees. Using the law of sines we can solve for AC: $\frac{x}{\sin(45^o)} = \frac{10}{\sin(30^o)}$, so $x = \frac{\frac{\sqrt{2}}{2}(10)}{\frac{1}{2}} = 10\sqrt{2}$, which rounded to the nearest whole number gives 14 meters.

5-2 Triangle SCL has side lengths $SC = 9$, $SL = 6$, and $CL = 11$. The line segment $\overline{SL}$ is extended to a point N, with L between S and N, in such a way that triangle SCN is isosceles. $CN = a\sqrt{b}$ where a and b are integers and b has no perfect square factors greater than 1. Find $b - 2a$.
[Answer: 38]

We can solve for the cosine of angle S using $11^2 = 6^2 + 9^2 - 2(6)(9)\cos(\angle C)$ to get $\cos(\angle C) = -\frac{1}{27}$. It follows that $CN^2 = 9^2 + 9^2 - 2(9)(9)\left(-\frac{1}{27}\right) = 168$, so $CN = 2\sqrt{42}$, making the desired quantity $42 - 2(2) = 38$.

5-3 Kite *ABCD* has right angles at *B* and *D*. The smallest angle in the kite is at *A*, and the measure of angle *A* is α, where $\cos\alpha = \frac{1}{8}$. The ratio of one of the shorter side lengths of the kite to one of its longer side lengths is $\frac{\sqrt{a}}{b}$, where a and b are integers and a has no perfect square factors larger than 1. Find $2a - b$.
[Answer: 11]

The largest and smallest angles of a kite are supplementary, making the cosine of the kite's largest angle $-\frac{1}{8}$. Letting the shorter side length of the kite equal x, the larger side length equal 1, and noting that the smallest and largest angles are vertex angles of isosceles triangles that share a base, we can set up $x^2 + x^2 - 2(x)(x)\left(-\frac{1}{8}\right) = 1^2 + 1^2 - 2(1)(1)\left(\frac{1}{8}\right)$, or $\frac{9}{4}x^2 = \frac{7}{4}$, making the desired ratio $x = \frac{\sqrt{7}}{3}$, and our desired value $2(7) - 3 = 11$.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 2

Round 6: Equations of Lines

6-1 Line l_1 passes through the origin and has a slope of $\frac{3}{4}$. Line l_2 is perpendicular to l_1 and intersects l_1 at a distance of 10 units from the origin in quadrant I. If line l_2 has an equation in slope-intercept form of $y = mx + b$, find the value of $b - m$.
[Answer: 18]

The equation of line l_1 is $y = \frac{3}{4}x$. We can see through inspection or by solving $(4a)^2 + (3a)^2 = 100$ to find that the point of intersection has to be $(8,6)$. We can find the equation of l_2 using $6 = -\frac{4}{3}(8) + b$, giving $b = \frac{50}{3}$. The desired value is therefore $\frac{50}{3} + \frac{4}{3} = 18$.

6-2 A line with the equation $y = \frac{7}{3}x + a$ can be represented parametrically by the equations $x = 3t + b$ and $y = 7t + 11$, where a and b constants. What is the value of $3a + 7b$?
[Answer: 33]

Letting $7t + 11 = \frac{7}{3}(3t + b) + a = 7t + \frac{7}{3}b + a$, we have $11 = a + \frac{7}{3}b$, so $3a + 7b = 33$.

6-3 Two perpendicular lines, one with a y-intercept of 17 and the other with a y-intercept of 4, intersect at the point $(6, k)$ where $k > 9$. What is the value of k?
[Answer: 13]

Letting the slope of the first line be m, we have the equation $6m + 17 = -\frac{6}{m} + 4$, giving an equation $6m^2 + 13m + 6 = 0$. Solving this quadratic yields two possible values of m: $-\frac{3}{2}$ or $-\frac{2}{3}$. Letting m equal either slope gives values of k of 8 or 13 respectively, but since $k > 9$, the desired value must be 13.

FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 2

Team Round

T-1 Let $f(x)$ be the number of positive integers less than or equal to x that are divisible by neither 6 nor 15. As x grows without bound, the value $\frac{f(x)}{x}$ approaches a rational number $\frac{p}{q}$, where p and q are positive integers with no common factors greater than 1. What is $p^2 + q^2$? [Answer: 41]

As x grows without bound, the value of $\frac{f(x)}{x}$ will approach the general proportion of integers less than or equal to x divisible by neither 6 nor 15. Note that $\frac{5}{6}$ of the integers will not be divisible by 6. From this we need to subtract the proportion of numbers remaining divisible by 15, which constitutes all odd multiples of 15 (since all even multiples of 15 are also multiples of 6 and therefore already accounted for), and all odd multiples of 15 comprise $\frac{1}{30}$ of the values less than or equal to x. Therefore, the proportion is $\frac{5}{6} - \frac{1}{30} = \frac{4}{5}$, making the desired quantity $4^2 + 5^2 = 41$.

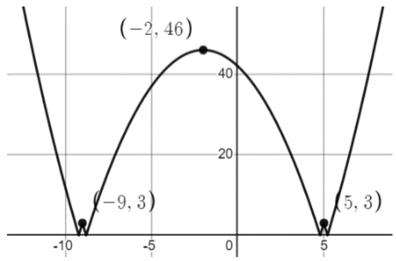
T-2 A fourth degree polynomial $f(x)$ has the properties that its coefficients are real, the quartic coefficient is 1, $f(i) = 0$, dividing $f(x)$ by $x - 2$ produces a remainder of 50, and dividing $f(x)$ by $x + 1$ produces a remainder of -10. Find the value of $f(3)$. [Answer: 190]

Given the conditions, we can assume that $f(x) = (x^2 + 1)(x^2 + ax + b)$, since $-i$ must be another 0. From here, we know $f(2) = 50$ and $f(-1) = -10$. Therefore $(5)(2a + b + 4) = 50$, so $2a + b = 6$, and $(2)(-a + b + 1) = -10$, so $-a + b = -6$. We can solve this system to find $a = 4$ and $b = -2$. Therefore $f(3) = (3^2 + 1)(3^2 + 4(3) - 2) = 10(19) = 190$.

T-3 Quadrilateral $QUAD$ is inscribed in a circle. Both $\overline{QA}$ and $\overline{UD}$ are diameters of the circle. If the perimeter of $QUAD$ is 48 and the area of $QUAD$ is 67.5, then the area of the circle is $\frac{a}{b}\pi$ where a and b are integers with no common factors greater than 1. Find $a - b$. [Answer: 437]

Since the diagonals of $QUAD$ are congruent and bisect each other, $QUAD$ must be a rectangle. Let the dimensions of $QUAD$ be x and y. We have $x + y = 24$ and $xy = 67.5$. This means $x^2 + y^2 = (x + y)^2 - 2xy = 24^2 - 2(67.5) = 576 - 135 = 441$, so $x^2 + y^2 = 21^2$. Therefore, the diameter of the circle is 21 and the radius is $\frac{21}{2}$, making the area $\frac{441}{4}\pi$ and our desired quantity $441 - 4 = 437$.

T-4 The equation $||x+9||x-5|-3| = k$, where k is a positive integer, has exactly three solutions when $k = a$ and exactly six solutions when $k = b$. Find the value of $|a-b|$.
[Answer: 43]



See the diagram. One approach to this problem is to consider the graphical transformations caused by the different absolute value calculations. The quantity $|x+9||x-5|$ is the absolute value of the quadratic $(x+9)(x-5)$. The latter has a vertex located at $(-2,-49)$, so making the former a quadratic whose negative values are reflected across the x-axis, with two minima at $(-9,0)$ and $(5,0)$ and a relative maximum at the reflected vertex whose coordinates are $(-2,49)$. The quantity $|x+9||x-5|-3$ shifts this graph down three units, giving the graph 4 x-intercepts, 2 relative minima with y-values of -3, and a relative maximum of the reflected and shifted vertex, $(-2,46)$. Finally, all negative values of this graph are reflected across the x −axis again. There are now six points where the y −value is 3 (the location of the reflected former minima) and three points at $y = 43$, making the desired value $|46-3| = 43$.

T-5 Consider triangle ABC with $AB = 3$, $AC = 7$, and $BC = 8$. If r is equal to the ratio of the sine of the smallest angle to the sine of the largest angle and t is the least numerical value of $\tan(A)$, $\tan(B)$, and $\tan(C)$, find the value of rt^2.
[Answer: 18]

Note that A is the largest angle and C is the smallest angle. Since $\frac{8}{\sin(A)} = \frac{3}{\sin(C)}$, we have $\frac{\sin(C)}{\sin(A)} = \frac{3}{8}$. Also, we note that $3^2 + 7^2 < 8^2$, making A an obtuse angle and also giving it the angle whose tangent has the least numerical value, since it is negative. Using the law of cosines, $\cos(A) = \frac{64-49-9}{-2(7)(3)} = -\frac{1}{7}$. Since $\tan^2(A) = \sec^2(A) - 1$, $\tan^2(A) = 49 - 1 = 48$, and the desired value is $\left(\frac{3}{8}\right)(48) = 18$.

T-6 Consider lines l_1 and l_2, where $l_1 \perp l_2$ and line l_1 has a positive slope. The lines intersect on the positive y-axis at point A, making a right triangle with the x-intercept of line l_1, B, and the x-intercept of line l_2, C. The y-axis intersects the hypotenuse of ABC at point D, and $CD = 4BD$. If the area of ABC is 245 square units, then the equation of l_2 can be written as $ax + by = c$ where a, b and c are positive integers that have no factors common among all three of them greater than 1. Find the value of $3a + 9b + c$.
[Answer: 49]

Triangle ABC is a right triangle with right angle A, and the altitude from A to the hypotenuse of the triangle lies on the y-axis. Setting $BD = x$, by the law of Pythagoras, $(4x)(x) = 4x^2 = (AD)^2$, so $AD = 2x$. The area of the triangle in terms of x is therefore $\frac{1}{2}(5x)(2x) = 5x^2 = 245$, so $x = 7$. This means that line l_2 has a slope of $-\frac{2x}{4x} = -\frac{1}{2}$ and a

y-intercept of $(0,14)$. Therefore, the equation of l_2 is $y = -\frac{1}{2}x + 14$, or in standard form, $x + 2y = 28$. This makes the desired value $3(1) + 9(2) + 28 = 49$.