FAIRFIELD COUNTY MATH LEAGUE 2022–2023
Match 1

Round 1: Percentages

1-1 I'm thinking of a three digit number, all of whose digits are nonzero. The hundreds digit is 75% of the tens digit, and the ones digit is 20% less than the sum of the tens digit and twice the hundreds digit. What is the number?
[Answer: 348]

Solution
Based on the first statement, the first three digits could be 3 and 4 or 6 or 8. However, while 20% less than $2(3) + 4$ is 8, 20% less than $2(6) + 8$ is 16, which is not a single digit number. Therefore the correct answer is 348.

1-2 How many positive integers have the property that increasing them by $66\frac{2}{3}\%$ and then decreasing that result by 65% produces a value greater than 100 but less than 1000?
[Answer: 1543]

Solution
Increasing a quantity by $66\frac{2}{3}\%$ is equivalent to multiplying it by $\frac{5}{3}$, and then decreasing that result by 65% is equivalent to multiplying it by $\frac{7}{20}$, making the net transformation multiplying by $\frac{35}{60}$ or $\frac{7}{12}$. Setting up $100 < \frac{7}{12}x < 1000$ yields a solution inequality of $171\frac{3}{7} < x < 1714\frac{2}{7}$, so the largest value that works is 1714 and the smallest that works is 172. The total amount of numbers is then $1714 - 171 = 1543$.

1-3 There is a positive number x such that increasing x by $x\%$ is equivalent to doubling the sum of x and $\frac{3}{8}$. If x can be written in simplest radical form as $a + b\sqrt{c}$, find the value of $2a - b + c$.
[Answer: 198]

Solution
Written as an equation, this becomes $\left(1 + \frac{x}{100}\right)x = 2\left(x + \frac{3}{8}\right)$. Expanding and writing this equation in standard form makes $x^2 - 100x - 75 = 0$. Solving this with the quadratic formula yields $x = \frac{100\pm\sqrt{10300}}{2} = \frac{100\pm10\sqrt{103}}{2} = 50 \pm 5\sqrt{103}$. Since the negative case is extraneous, this gives $x = 50 + 5\sqrt{103}$, so the desired result is $2(50) - 5 + 103 = 198$.

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Round 2: Solving Equations

2-1 Solve for x: $3x - 5\left(x + 7\big(2x - 6(x+1)\big)\right) = 6(25x+1)$
[Answer: 17]

Solution
Performing the necessary multiplications and simplifications yields $138x + 210 = 150x + 6$, which makes $12x = 204$, yielding $x = 17$.

2-2 The ordered pair $(2,7)$ is one ordered pair (x, y) that solves the equation $ax^2 - by = 1$ where a and b are positive integers less than 100. What is the largest possible value of b?
[Answer: 53]

Solution
Substituting yields $4a - 7b = 1$. We can see by inspection that $a = 2$ and $b = 1$ satisfies the equation. Adding 7 to a until we reach the largest possible value of $a < 100$ gives $2 + 13(7) = 93$, so the largest possible value of b is $1 + 13(4) = 53$.

2-3 Find the sum of the squares of all real solutions to the equation $\frac{3x+12}{x-2} + 3x^2 - 2 = \frac{3x^3-6}{x-2} + 5x$.
[Answer: 1]

Solution
Multiplying every term by $x - 2$ yields $3x + 12 + (3x^2 - 2)(x - 2) = 3x^3 - 6 + 5x(x-2)$, which simplifies to $3x^3 - 6x^2 + x + 16 = 3x^3 + 5x^2 - 10x - 6$, which in standard form becomes $x^2 - x - 2 = 0$. This yields solutions of $x = 2$ or $x = -1$, but $x = 2$ is extraneous, making the desired result $(-1)^2 = 1$.

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Round 3: Triangles and Quadrilaterals

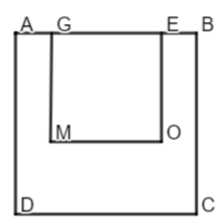
3-1 If a is the measure of one angle of an equilateral triangle in degrees, b is the largest possible integer degree measure of a base angle of an isosceles triangle, and c is the degree measure of the largest angle in a right triangle, find $2a + 3b + c$.
[Answer: 477]

Solution
Given that the measure of one angle of an equilateral triangle is 60 degrees, the largest possible integer degree measure of a base angle of an isosceles triangle is 89 degrees, and the largest angle of a right triangle is the right angle, the desired value is $2(60) + 3(89) + 90 = 477$.

3-2 A rhombus has diagonals of integer length and a perimeter of 100. What is the largest possible area of the rhombus?
[Answer: 600]

Solution
Since a rhombus can be broken up into four right triangles, this rhombus could be broken into four right triangles, each with a hypotenuse of length 25. In order for the diagonal lengths to be integers, the lengths of legs of right triangles must be integers as well. The three possible lengths would be 15, 20, and 25 or 7, 24, and 25. This would give diagonal lengths of either 30 and 40 units or 14 and 48 units, giving two possible areas of $.5(30)(40) = 600$ square units or $.5(14)(48) = 336$ square units, so the largest possible value is 600.

3-3 Consider the diagram to the right (not drawn to scale) showing squares $ABCD$ and $GEOM$, with points G and E on $\overline{AB}$ such that $AG = EB$. If the ratio of the areas of the squares is $\frac{3}{8}$, then the ratio of the area of trapezoid $DMOC$ to the area of square $ABCD$ is $\frac{p}{q}$ where p and q are relatively prime integers. Find $p + q$.
[Answer: 21]



Solution
Let $AG = EB = y$ and $GE = x$. Then the length of one side of the larger square is then $x + 2y$. The area of trapezoid $AGMD$ would then be $\frac{1}{2}(y)(x + x + 2y) = y(x + y)$, which is equal to the area of trapezoid $BEOC$. The area of trapezoid $DMOC$ is then $\frac{1}{2}(2y)(x + x + 2y) = 2y(x + y)$, or twice that of the smaller trapezoids. This means that the area of $DMOC$ is exactly half of the remaining area not taken up by square $GEOM$, so half of $\frac{5}{8}$ is $\frac{5}{16}$, making the desired quantity $5 + 16 = 21$.

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Round 4: Systems of Equations

4-1 At Clog O' Burger, 6 burgers and 4 orders of fries is $46.50, and 4 burgers and 6 orders of fries is $43.50. What is the total cost of 2 burgers and 2 orders of fries at Clog O' Burger in dollars?
[Answer: 18]

Solution
Let b be the price of a burger and let f be the price of an order of fries. From the problem we have $6b + 4f = 46.50$ and $4b + 6f = 43.50$, so $10b + 10f = 90$, or $b + f = 9$. Therefore, the price of 2 burgers and 2 orders of fries in dollars is $2(9) = 18$.

4-2 Given the system $\begin{cases} \frac{8}{x} + \frac{3}{y} = 24 \\ \frac{2}{x} - \frac{9}{y} = -7 \end{cases}$, find the value of $10xy$.
[Answer: 3]

Solution
Adding three times the first equation to the second equations yields $\frac{26}{x} = 65$, so $x = \frac{26}{65} = \frac{2}{5}$. Substituting this back into either equation yields $y = \frac{3}{4}$. Therefore $10xy = 10\left(\frac{2}{5}\right)\left(\frac{3}{4}\right) = 3$.

4-3 The graphs of $x^2 - y^2 + 3x + 4y - 31 = 0$ and $x + y = 10$ intersect at the point (a, b). Find the value of $20a + b$.
[Answer: 101]

Solution
We can rewrite the first equation as $(x + y)(x - y) + 3(x + y) + y - 31 = 0$, and substituting $x + y = 10$ we can produce $10(x - y) + 30 + y - 31 = 0$, or $10x - 9y = 1$. Adding $10(x + y) = 100$ to both sides yields $20x + y = 101$, providing our answer of $20a + b = 101$.

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Round 5: Right Triangles

5-1 A right triangle has the properties that the tangent of one of its acute angles has a value of $\frac{4}{5}$ and the area of the triangle is 50 square units. What is the square of the length of the hypotenuse?
[Answer: 205]

Solution
If the triangle has legs of length a and b, then $\frac{a}{b} = \frac{4}{5}$ and $\frac{1}{2}ab = 50$. This means $\frac{1}{2}\left(\frac{4}{5}b\right)b = 50$, so $\frac{2}{5}b^2 = 50$, so $b^2 = 125$. Since $\frac{a^2}{125} = \frac{16}{25}$, $a^2 = 80$, so $c^2 = 80 + 125 = 205$.

5-2 A balloon is flying between two points on level ground, point A and point B. If the tangent of the angle of elevation to the balloon from point A is $\frac{2}{3}$ and the tangent of the angle of elevation to the balloon from point B is $\frac{3}{8}$ and the ground distance between points A and B is 900 feet, find the height above the ground of the balloon in feet.
[Answer: 216]

Solution
We have $\tan(A) = \frac{2}{3} = \frac{6}{9}$ and $\tan(B) = \frac{3}{8} = \frac{6}{16}$. This means the ratio of the ground height of the balloon to the distance between points A and B is $\frac{6}{25}$. Setting $\frac{6}{25} = \frac{x}{900}$ yields $x = 216$.

5-3 A right triangle has all integer side lengths. One of the legs has a length of 48. What is the largest possible length of the hypotenuse?
[Answer: 577]

Solution
One approach to this is set $48^2 = c^2 - b^2$, which means $2304 = (c+b)(c-b)$. The maximum value of c will occur where $c - b$ is at a minimum, which must be 2 (setting $c - b = 1$ produces non-integer values of b and c). This means $c + b = 1152$. Solving the system gives a value of $c = 577$.

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Round 6: Coordinate Geometry

6-1 The point $(5,-8)$ is rotated 90 degrees counterclockwise about the origin, then shifted up 7 units, and then reflected across the line $y=x$ to produce the new point with coordinates (p,q). Find the value of $2p+3q$.
[Answer: 48]

Solution
Rotating $(5,-8)$ ninety degrees counterclockwise about the origin transforms it to the point $(8,5)$. Shifting it up 7 units makes $(8,12)$, and then reflected across $y=x$ becomes $(12,8)$. This makes the desired quantity $2(12)+3(8)=48$.

6-2 The point $(2,7)$ is exactly three units away from the point $(2,10)$ which lies on the line $y=2x+6$. There is one other point on the line $y=2x+6$ that lies three units away from $(2,7)$. If its coordinates are (a,b), find the value of $a+2b$.
[Answer: 10]

Solution
Considering the point $(a,2a+6)$ and setting it three units away from $(2,7)$ gives the equation $(a-2)^2+(2a+6-7)^2=9$. Expanding and writing this equation in standard form yields $5a^2-8a-4=0$, which factors into $(a-2)(5a+2)=0$. Since $a=2$ is an already known solution, we want $a=-\frac{2}{5}$, which gives a y-value of $\frac{26}{5}$. The desired value is then $-\frac{2}{5}+2\left(\frac{26}{5}\right)=10$.

6-3 The points $(2,3)$, $(6,7)$, and $(4,1)$ all lie on a circle with center (h,k) and radius R. Find $\frac{kR^2}{h}$.
[Answer: 8]

Solution
One way to find the equation of the circle is to construct perpendicular bisectors between two pairs of points, since these will intersect at the center of the circle. The slope of the line through $(2,3)$ and $6,7)$ is 1 and the midpoint is $(4,5)$, making the equation of the perpendicular bisector $y=-x+9$. The slope between $(2,3)$ and $(4,1)$ is -1 and the midpoint is $(3,2)$, making the equation of the perpendicular bisector $y=x-1$. Setting $-x+9=x-1$ gives $x=5$ and thus $y=4$. The equation of the circle is then $(x-5)^2+(y-4)^2=R^2$, and substituting any of the given points for x and y gives $R^2=10$, so our desired quantity $\frac{4(10)}{5}=8$.

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Team Round

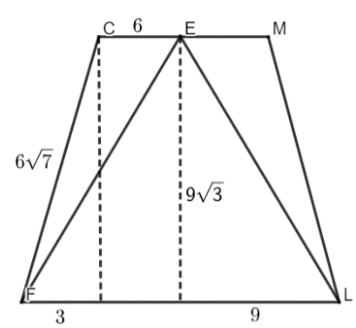
T-1 Two positive integers a and b have the property that $a\%$ of b is equal to the sum of 60% of a and 30% of b. What is the second largest possible value of b?
[Answer: 960]

Solution
Written as an equation, this becomes $\frac{ab}{100} = \frac{60}{100}a + \frac{30}{100}b$, or $ab = 60a + 30b$. If we isolate b we get $b = \frac{60a}{a-30}$. The largest possible value of b occurs when the denominator is 1, and the second largest value of b occurs when the denominator is 2. This happens when $a = 32$, so $b = \frac{60(32)}{2} = 960$.

T-2 The equation $\frac{5}{x+1} + \frac{1}{x+25} + 2 = \frac{2x}{x-2}$ has solutions x_1 and x_2, where $x_1 > x_2$. Find $x_1 - x_2$.
[Answer: 27]

Solution
Multiplying every term by $(x+1)(x+25)(x-2)$ yields $5(x+25)(x-2) + (x+1)(x-2) + 2(x+1)(x+25)(x-2) = 2x(x+1)(x+25)$, which after expansion makes $5x^2 + 115x - 250 + x^2 - x - 2 + 2x^3 + 48x^2 - 54x - 100 = 2x^3 + 52x^2 + 50x$. In standard form this makes $2x^2 + 10x - 352 = 0$, or $x^2 + 5x - 176 = 0$. This factors into $(x+16)(x-11) = 0$, making solutions of $x_1 = 11$ and $x_2 = -16$, so $x_1 - x_2 = 27$.

T-3 Consider isosceles trapezoid $FCML$ with bases $\overline{FL}$ and $\overline{CM}$ and point E on $\overline{CM}$ such that the inscribed triangle ELF is equilateral. If $FL = 18$ and the triangle comprises 60% of the trapezoid's total area, then the perimeter of $FCML$ can be written as $x + y\sqrt{z}$ where $x. y,$ and z are positive integers and z has no perfect square factors greater than 1.
Find $x + y + z$.
[Answer: 49]



Solution
See the diagram. If $FL = 18$, then the height of the trapezoid is $9\sqrt{3}$. If the triangle takes up 60% of the trapezoid's area, then the ratio of the bottom base to the top base is $3:2$, making the top base 12 units. The lengths of the legs can be computed using the Pythagorean theorem as $\sqrt{3^2 + \left(9\sqrt{3}\right)^2} = \sqrt{252} = 6\sqrt{7}$. Therefore the total perimeter is $30 + 12\sqrt{7}$, making the desired result $30 + 12 + 7 = 49$.

T-4 If A and B are positive numbers such that the system $\begin{cases} Ax + By = 6 \\ 9x + Ay = 10 \end{cases}$ has infinite solutions for (x, y), then B can be written in simplest form as $\frac{p}{q}$ where p and q are relatively prime integers. Find the value of $p + q$.
[Answer: 106]

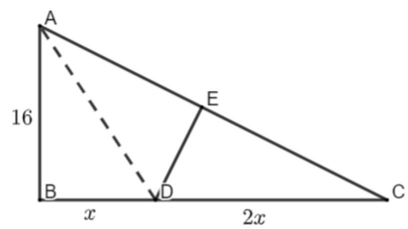
Solution

Knowing the system has infinite solutions tells us $\frac{A}{9} = \frac{B}{A} = \frac{6}{10}$. Since $A^2 = 9B$, we have $A = 3\sqrt{B}$. Therefore $\frac{3\sqrt{B}}{9} = \frac{3}{5}$, so $\sqrt{B} = \frac{9}{5}$, and $B = \frac{81}{25}$ and $81 + 25 = 106$.

T-5 Consider right triangle ABC with right angle B, D on $\overline{BC}$, and E on $\overline{AC}$ such that $\overline{DE} \perp \overline{AC}$. If $AB = 16$, $\tan(\angle CAB) = 3\tan(\angle DAB)$, and $\sin(\angle ACB) = \frac{8}{17}$, then $DE = \frac{x}{y}$ where x and y are integers with no common factors greater than 1. Find $x - y$.
[Answer: 143]

Solution

See the diagram (not necessarily drawn to scale). We can use the sine of angle C to find that $AC = 34$, which also tells us that $BC = 30$. We also know that $CB = 3DB$, which means $CD = 2DB$ and therefore $CD = 20$. We can then use similarity to set up $\frac{8}{17} = \frac{DE}{20}$, so $DE = \frac{160}{17}$, making the desired result $160 - 17 = 143$.



T-6 A line with equation $Ax + By = 120$ where A and B are integers has the properties that it contains the point (3,5) and it crosses the $x-$ and $y-$axes respectively at $(c, 0)$ and $(0, d)$ where c and d are positive integers. Find the sum of all possible values of B.
[Answer: 33]

Solution

Substituting $(3, 5)$ for x and y yields $3A + 5B = 120$. We can see by inspection that $A = 40$ and $B = 0$ solves the equation but does not produce a line with 2 distinct positive integer intercepts. We can continue to get values for (A, B) that solve the equation by decreasing A by 5 and increasing B by 3, yielding potential solutions of $(35, 3)$, $(30, 6)$, $(25, 9)$, $(20, 12)$, $(15, 15)$, $(10, 18)$, $(5, 21)$, and $(0, 24)$. However the only pairs such that $\frac{120}{A}$ and $\frac{120}{B}$ are both positive integers are $(30, 6)$, $(20, 12)$, and $(15, 15)$, so the sum of all possible values of B is $6 + 12 + 15 = 33$.